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INEQUALITIES FOR RESIDUALS OF POWER SERIES: A REVIEW

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Dedicated to the memory of Professor Dragoslav S. Mitrinović

Let f be a function that can be represented as a sum of convergent Maclaurin series and let $I_n(x) = f(x) - \sum_{k=n}^{+\infty} f^{(k)}(0)\frac{x^k}{k!}$. We give a survey of inequalities related to I_n for $f(x) = e^x$ and some recent results regarding other functions.

1. INTRODUCTION

In the celebrated monograph *Analytic Inequalities* by D. S. Mitrinović [**17**], there are several inequalities related to the residual of Maclaurin expansion for the exponential function:

$$I_n(x) = \sum_{k=n+1}^{+\infty} \frac{x^k}{k!}.$$

The appearance of [**17**] helped widespreading a general knowledge about these inequalities (as it was the case with many other types of inequalities), and a number of papers have been published since 1970, with their improvements and generalizations. A revival of interest in this topic occured in recent years, with several papers concerning I_n for exponential and other functions.

In this paper we are dealing with inequalities quoted in sections 3.6.3., 3.6.6., 3.6.7., 3.6.16. and 3.8.25. of [**17**] and their subsequent improvements and generalizations.

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2. RAMANUJAN'S PROBLEM

In 1911, S. Ramanujan posed the following problem [**22**]:

Show that, if x is a positive integer,

$$\frac{1}{2}\,e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^x}{x!}\theta(x), \tag{1}$$

where $\theta(x)$ lies between $1/2$ and $1/3$.

In terms of I_n, this problem can be formulated as

$$I_{n-1}(n) = \frac{1}{2}e^n + \frac{n^n}{n!}\theta \qquad \text{for } \frac{1}{3} < \theta < \frac{1}{2}.$$

Ramanujan offered a partial solution next year in [**23**]; a complete solution was presented by Szegö [**20**] and Watson [**24**] in 1928. In 1960, Karamata [**11**] revisited this problem and gave an elegant solution. Karamata also provided an asymptotic expansion for $\theta(x)$, as defined by (1).

In 1991, Merkle [**14**] noticed that Karamata's proof of (1) leads to a more general inequality

$$\frac{e^x}{\Gamma(x)}\int_0^x e^{-t}t^{x-1}\mathrm{d}t = \frac{1}{2}\,e^x + \frac{x^x}{\Gamma(x)}\,\theta, \qquad \frac{1}{3} < \theta < \frac{1}{2}, \quad x > 0,$$

which, in turn, leads to a simple proof of the fact that the median of a chi squared distribution with α degrees of freedom lies in the interval $(\alpha - 1, \alpha)$ for all $\alpha > 1$.

3. SOME BOUNDS FOR I_n.

From the Taylor's formula it simply follows that

$$\frac{x^{n+1}}{(n+1)!} \le I_n(x) \le \frac{x^{n+1}}{(n+1)!}\,e^x, \qquad x \ge 0. \tag{2}$$

In the section 3.8.25 of [**17**], the following inequality for complex argument is given

$$|I_n(z)| \le \frac{|z|^{n+1}}{(n+1)!}e^{|z|}, \qquad z \in \mathbf{C}, \tag{3}$$

with a reference to Garnir's book [**9**]. This can be easily obtained from (2) by the triangle inequality. In the same section there is a note attributed to P. R. Beesack, with the partial improvement of (3) to

$$|I_n(z)| \le \frac{(n+2)|z|^{n+1}}{(n+1)!(n+2-|z|)}, \qquad |z| < n+2. \tag{4}$$

Martić [**12**] in 1975. gives the inequality

$$|I_n(z)| \le \frac{|z|^k}{(n+1)n\cdots(n-k+2)} I_{n-k}(|z|), \quad z\in \mathbf{C},\ k=1,2,\dots,n+1 \tag{5}$$

and he shows that (5) is sharper than (3) for $|z| > ((n+1-k)!)^{1/(n+1-k)}$.

In [**12**] there is also the inequality

$$I_n(x) < \frac{x^k}{(n+1)n\cdots(n-k+2)} I_{n-k}(x), \qquad x>0,\ k=1,2,\dots n+1,$$

which is an improvement over a crude bound from [**18**]:

$$I_n(x) < \frac{x}{n}e^x, \qquad x>0.$$

4. KESAVA MENON'S INEQUALITY AND ITS OMPROVEMENTS

In 1943, Kesava Menon [**13**] gave the inequality quoted in 3.6.4. of [**17**]:

$$\frac{I_{n-1}(x)I_{n+1}(x)}{{I_n}^2(x)} > \frac{1}{2}, \qquad x>0. \tag{6}$$

Using the equality

$$I_n(x) = I_{n+1}(x) + \frac{x^{n+1}}{(n+1)!},$$

it can be shown that (6) is equivalent to

$${I_n}^2(x) - 2I_n(x)\frac{x^n}{n!}\left(1-\frac{x}{n+1}\right) - \frac{x^{2n+1}}{n!(n+1)!} > 0,$$

wherefrom it follows that

$$I_n(x) > \frac{x^{n+1}}{(n+1)!}\left(\frac{x}{n+2} + \sqrt{\frac{x^2}{(n+1)^2}+1}\right). \tag{7}$$

Alzer [**1**] shows in 1991. that (6) can be improved to

$$\frac{I_{n-1}(x)I_{n+1}(x)}{{I_n}^2(x)} > \frac{n+1}{n+2}, \qquad x>0. \tag{8}$$

From the Taylor's formula we have that

$$I_n(x) = e^{\xi_n(x)}\frac{x^{n+1}}{(n+1)!}, \qquad 0<\xi_n(x)<x \tag{9}$$

and

$$\lim_{x\to 0}\frac{I_{n-1}(x)I_{n+1}(x)}{{I_n}^2(x)}=\frac{n+1}{n+2}.$$

Therefore, the constant $(n+1)/(n+2)$ in (8) is the best possible.

Merkle [**15**] showed that (8) is equivalent to

$$I_n(x)>\frac{(n+2)x^n}{2n!}\left(\frac{x}{n+1}-1+\sqrt{\left(\frac{x}{n+1}-\frac{n}{n+2}\right)^2+\frac{4(n+1)}{(n+2)^2}}\right),\qquad x>0. \tag{10}$$

The inequality (10) is very sharp. If we denote by $D(x,n)$ the difference between left and right hand side in (10) and

$$R(x,n)=\frac{D(x,n)}{e^x},$$

then it can be shown that $R(x,n)$ tends to zero as either $x\to 0$ or $n\to+\infty$. For example, $R(1,5)=7.5\cdot 10^{-7}$, $R(0.5,5)=3.7\cdot 10^{-9}$, etc.

The form of inequalities (6) or (8) suggests a logarithmic covexity. Indeed, (8) is equivalent to the statement that the mapping $n\mapsto(n+1)!\,I_n(x)$ is log-convex for $x\in[0,+\infty)$ (see [**2**] or [**16**]). From this it follows that the mapping $x\mapsto\xi_n(x)$, with $\xi_n(x)$ as in (9), is convex, hence

$$\xi_n(x)\le\frac{j}{j+k}\,\xi_{n-k}(x)+\frac{k}{j+k}\,\xi_{n+j}(x),$$

for any $x>0$, $n,j,k\in\mathbf{N}$.

In [**15**] it is shown that the mapping $n\mapsto I_n(x)$ is log-concave on $[0,+\infty)$, i.e.,

$$\frac{I_{n-1}(x)I_{n+1}(x)}{{I_n}^2(x)}<1,\qquad x>0. \tag{11}$$

From (8) and (11) it follows that

$$\lim_{n\to+\infty}\frac{I_{n-1}(x)I_{n+1}(x)}{{I_n}^2(x)}=1.$$

The inequality (11) implies the inequality

$$I_n(x)<\frac{x^{n+1}}{n!(n-x+1)},\qquad 0<x<n+1.$$

Incidentally, this inequality was obtained by Karamata in [**10**] by other means. However, this is a weaker result than (4), indicating that 1 in (11) could probably be replaced by a smaller constant.

Logarithmic convexity was employed for a further generalization by Alzer, Brenner and Ruehr [**2**]. They considered the expression

$$D_{n,k}(x)=(n-k+1)!(n+k+1)!I_{n+k}(x)-((n+1)!)^2{I_n}^2(x).$$

They showed that for a fixed n and $x > 0$, the sequence $\{D_{n,k}(x)\}$ is nonincreasing and convex in k, for $0 \le k < n$ and for a fixed k, it is logarithmically convex in n, for $n > k \ge 0$.

Some of the results of this section follow from Theorem 4 in FINK's paper [**8**]. It was shown there that if f is a function defined on $[0,T]$ such that for $t \in [0,T)$

$$f(0) = f'(0) = \cdots = f^{(N-1)}(0) = 0, \ f^{(N)}(0) \ge 0, \text{ and } f^{(N+1)}(t) \ge 0, \tag{12}$$

then

$$f^{(\alpha_1)}(t) \cdots f^{(\alpha_r)}(t) \ge K(\alpha,\beta,N) f^{(\beta_1)}(t) \cdots f^{(\beta_r)}(t), \tag{13}$$

for all $t \in [0,T]$ and for all $\alpha = (\alpha_1, \dots, \alpha_r)$ and $\beta = (\beta_1, \dots, \beta_r)$ such that

$$\sum_{i=1}^{k} \alpha_i \ge \sum_{i=1}^{k} \beta_i, \quad (1 \le k \le r-1), \qquad \sum_{i=1}^{r} \alpha_i = \sum_{i=1}^{r} \beta_i,$$

where α_i and β_i are nonegative integers and

$$K(\alpha,\beta,N) = \prod_{i=1}^{r} \frac{(N-\beta_i)!}{(N-\alpha_i)!}.$$

It is easy to show that the function $t \mapsto I_n(t)$ satisfies conditions (12) with $N = n+1$; moreover, $I_n^{(k)} = I_{n-k}$ for $0 \le k \le n$. By taking, for example, $f(t) = I_{n+1}(t)$, $r = 2$, $\alpha = (2,0)$, $\beta = (1,1)$, we get the inequality (8).

5. EXTENSIONS OF KESAVA MENON'S INEQUALITY

In this section we will consider the residual of MACLAURIN expansion for an arbitrary infinitelly differentiable function f with all derivatives positive and with a convergent MACLAURIN series on $(-R,R)$:

$$I_n(x) = \sum_{k=n+1}^{+\infty} f^{(k)}(0) \frac{x^k}{k!}.$$

An analogue of (8) for such a function would be

$$\frac{I_{n-1}(x) I_{n+1}(x)}{I_n^2(x)} > \frac{n+1}{n+2}, \qquad x \in (0,R). \tag{14}$$

In the paper by MERKLE and VASIĆ [**16**], the inequality (14) is proved to hold for functions f with positive derivatives of all orders at zero and such that the mapping $n \mapsto f^{(n)}(0)$ is log-convex. Starting from that fact, they proved that (8) holds if the mapping $n \mapsto f^{(n)}(0)$ is logarithmically convex.

It can be shown that

$$\lim_{x\to 0}\frac{I_{n-1}(x)I_{n+1}(x)}{{I_n}^2(x)}=\frac{f^{(n)}(0)f^{(n+2)}(0)}{(f^{(n+1)}(0))^2}\cdot\frac{n+1}{n+2}, \tag{15}$$

and this fact leads to the investigation of inequality

$$\frac{I_{n-1}(x)I_{n+1}(x)}{{I_n}^2(x)}\geq\frac{f^{(n)}(0)f^{(n+2)}(0)}{(f^{(n+1)}(0))^2}\cdot\frac{n+1}{n+2},\qquad x\in(0,R). \tag{16}$$

Clearly, if (16) holds for a function f, then by (15), the constant at the right hand side of (16) can not be replaced by a smaller constant.

The inequality (16) was investigated by Dilcher [**6**], who proved that it holds for all functions with positive derivatives at zero, such that the mapping $n\mapsto\gamma_n(x)$ is decreasing and convex, where

$$\gamma_n(x)=\frac{f^{(n+1)}(0)}{(n+1)f^{(n)}(0)},\qquad n=0,1,\dots$$

Merkle and Vasić proved in [**16**] that (16) holds if the mapping $n\mapsto\gamma_n$ is logarithmically convex.

Let $a_n=f^{(n)}(0)/n!$ and let r,s be natural numbers such that

$$\frac{a_{n+1}}{(a_{n+1-s})^{r/(s+r)}(a_{n+1+r})^{s/(s+r)}}\quad\text{is nonincreasing in } n. \tag{17}$$

Chen [**4**] proved that if (17) holds then

$$I_n(x)\leq\frac{a_{n+1}}{(a_{n+1-s})^{r/(s+r)}(a_{n+1+r})^{s/(s+r)}}(I_{n-s}(x))^{r/(s+r)}(I_{n+r}(x))^{s/(s+r)}, \tag{18}$$

for $x\in(0,R)$.

A special case $s=r=1$ of (17) yields the inequality (16).

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