Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 8 (1997), 114–119.

ON LOG-CONVEXITY OF A RATIO OF GAMMA FUNCTIONS

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We prove that the function $x \mapsto \Gamma(2x)/x\Gamma^2(x)$ is strictly log-convex and the function $x \mapsto \Gamma(2x)/\Gamma^2(x)$ is strictly log-concave on $x > 0$ and we present some consequences of these results.

1. INTRODUCTION

A positive function f is said to be logarithmically convex (or log-convex) if the function $x \mapsto \log f(x)$ is convex. It is well known that the logarithmic convexity is a fundamental property of the Gamma function [**2, 3**]. This paper continues our study [**7, 8, 9**] of convexity and Schur-convexity of functions related to the Gamma function. For a convenience, let us recall some basic facts about Schur-convexity. More details can be found in [**6**].

Given two vectors $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ of dimension n, we say that x is majorized by y if

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]} \quad \text{for } k = 1, 2, \dots, n-1 \quad \text{and} \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i,$$

where $(x_{[1]}, x_{[2]}, \dots, x_{[n]})$ is decreasing rearrangement of coordinates of x. If x is majorized by y, we write $x \prec y$. A function f of n variables is said to be Schur-convex on $A \subset \mathbf{R}^n$ if

$$x \prec y \Rightarrow f(x) \le f(y) \qquad \text{for each } x, y \in A. \tag{1}$$

If $x \prec y$ implies $f(x) < f(y)$ whenever $x, y \in A$ and x is not a permutation of y, we say that f is a strictly Schur-convex function.

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^{0}1991 Mathematics Subject Classification: 33A15, 26D20

Let g be a continuous nonnegative function defined on an interval $I \subset \mathbf{R}$. Then

$$f(x) = \prod_{i=1}^{n} g(x_i), \qquad x \in I^n$$

is Schur-convex (strictly Schur-convex), on I^n if and only if g is log-convex (strictly log-convex) on I.

2. CONVEXITY RESULTS

Theorem 1. *Define*

$$F(x) = \frac{\Gamma(2x)}{x\Gamma^2(x)}, \quad G(x) = \frac{\Gamma(2x)}{\Gamma^2(x)} \ . \tag{2}$$

For $x > 0$, the function F is strictly log-convex and the function G is strictly log-concave.

Proof. By the duplication formula for the Gamma function

$$\Gamma(2x) = (2\pi)^{-1/2} 2^{2x-1/2} \Gamma(x)\Gamma(x+1/2) \tag{3}$$

(see [**1**] for example) we have that

$$F(x) = (2\pi)^{-1/2} 2^{2x-1/2} \frac{\Gamma(x+1/2)}{\Gamma(x+1)}$$

and therefore

$$(\log F(x))'' = \sum_{k=0}^{+\infty} \left(\frac{1}{(x+1/2)^2} - \frac{1}{(x+1)^2} \right) > 0 \quad \text{for } x > 0. \tag{4}$$

So, the function F is strictly log-convex on $(0, +\infty)$.

The second derivative of the function $x \mapsto \log G(x)$ is

$$(\log G(x))'' = 4\Psi'(2x) - 2\Psi'(x), \tag{5}$$

where Ψ is the Digamma function and

$$\Psi'(x) = \sum_{k=0}^{+\infty} \frac{1}{(x+k)^2} \ .$$

By the duplication formula (3) we have

$$4\Psi'(2x) = \Psi'(x) + \Psi'(x+1/2),$$

and we find from (5) that

$$(\log G(x))'' = \Psi'(x+1/2) - \Psi'(x) < 0, \qquad (x > 0).$$

Hence, G is a strictly log-concave function on $x > 0$.

Corollary 1. *For $x > 0$ and $\beta \in [0,1]$ we have the following inequalities:*

$$\frac{\Gamma(2x+2\beta)}{\Gamma^2(x+\beta)} \le \frac{x+\beta}{x}\left(\frac{2(2x+1)}{x+1}\right)^{\beta}\frac{\Gamma(2x)}{\Gamma^2(x)}, \tag{6}$$

$$\frac{\Gamma(2x+2\beta)}{\Gamma^2(x+\beta)} \ge \left(\frac{2(2x+1)}{x}\right)^{\beta}\frac{\Gamma(2x)}{\Gamma^2(x)}. \tag{7}$$

The equality in both inequalities occurs if and only if $\beta = 0$ or $\beta = 1$. For $\beta > 1$, inequalities (6) and (7) hold with "$\le$" and "$\ge$" interchanged.

Proof. The first inequality follows from Jensen's inequality

$$\log F(x+\beta) \le (1-\beta)\log F(x) + \beta F(x+1) \tag{8}$$

for the log-convex function F defined in Theorem 1. The second inequality follows from

$$\log G(x+\beta) \ge (1-\beta)\log G(x) + \beta G(x+1), \tag{9}$$

where G is the log-concave function defined in Theorem 1. The statement about equality is a consequence of the strict convexity of F and the strict concavity of G. If $\beta > 1$ then Jensen's inequalities (8) and (9) are reversed, and so (6) and (7) are also reversed. $\square$

In a special case $x = 1$, Corollary 1 yields an interesting double inequality

$$6^{\beta} < \frac{\Gamma(2(1+\beta))}{\Gamma^2(1+\beta)} < (1+\beta)3^{\beta} \qquad (0 < \beta < 1), \tag{10}$$

and the reversed inequality for $\beta > 1$.

Corollary 2. *The function*

$$F_1(x) = \log\Gamma(2x+1) - 2\log\Gamma(x+1)$$

is strictly convex on $x > -1/2$.

Proof. It is easy to see that $F_1(x)'' = (\log F(x))''$, where F is defined in Theorem 1. Then from (4) we conclude that F_1 is convex on $x > -1/2$.

Corollary 3. *For all $x \ge 1$ we have*

$$\frac{\Gamma(2x+1)}{\Gamma^2(x+1)} \ge 2^{x}, \tag{11}$$

with equality if and only if $x = 1$. If $0 \le x < 1$, then

$$\frac{\Gamma(2x+1)}{\Gamma^2(x+1)} \le 2^{x}, \tag{12}$$

with equality if and only if $x = 0$.

Proof. Inequality (11) is equivalent to

$$\varphi(x) = \log\Gamma(2x+1) - 2\log\Gamma(x+1) - x\log 2 > 0.$$

For $x \geq 0$ we have the following:

$$\varphi'(x) = 2\psi(2x+1) - 2\psi(x+1) - \log 2, \qquad \varphi''(x) = F_1''(x) > 0,$$

where ψ is the Digamma function and F_1 is the function defined in the statement of Corollary 2. The recurence relation $\psi(z+1) = \psi(z)+1/z$ yields $\varphi'(1) = 1-\log 2 > 0$ and by $\varphi''(x) > 0$ we have that $\varphi'(x) > 0$ for $x > 1$; therefore $\varphi(x) > \varphi(1) = 0$. On the other hand, since $\varphi(0) = 0$, $\varphi'(0) < 0$ and $\varphi''(x) > 0$, φ must be unimodal. Then from $\varphi(1) = 0$ it follows that $\varphi(x) < 0$ for $0 < x < 1$, which is equivalent to (12).

Corollary 4. *The function*

$$\phi(x) = \prod_{i=1}^{n} \frac{\Gamma(2x_i+1)}{\Gamma^2(x_i+1)}$$

is strictly Schur-convex on $x = (x_1, \ldots, x_n) \in (-1/2, +\infty)^n$.

Proof. By Corollary 2, the function $x \mapsto \Gamma(2x+1)/\Gamma^2(x+1)$ is strictly log-convex on $x > -1/2$ and therefore by a result mentioned in Section 1, the function ϕ is Schur-convex.

3. SOLUTION TO A PROBLEM POSED BY MITRINOVIĆ AND PEČARIĆ

The inequality (11) is not very sharp for large x. It can be improved in many ways. In fact, it is not difficult to show that from

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma^2((a+b)/2)} \geq \frac{a^a b^b}{((a+b)/2)^{a+b}}$$

(see [**4**]) with $a = 2x+1$ and $b = 1$ we obtain an inequality which is sharper than (11) for $x > 1+\varepsilon$ where ε can be determined numerically. However, the results of Section 2 might be interesting from another viewpoint. In [**11**] the following problem was proposed: Let $x_1, x_2, \ldots, x_r, x$ be nonnegative real numbers such that $x_1 + \cdots + x_r = x$. Prove (possibly under some additional conditions) that

$$\prod_{i=1}^{r} \frac{\Gamma^2(x_i+1)}{\Gamma(2x_i+1)} \leq \frac{1}{2^x}. \tag{13}$$

From (12) it follows that (13) is false if $0 < x_i < 1$ for all $i = 1, 2, \ldots r$. On the other hand, since

$$(\bar{x}, \bar{x}, \cdots, \bar{x}) \prec (x_1, x_2, \ldots, x_r) \qquad \bar{x} = \frac{x}{r} = \frac{x_1 + x_2 + \cdots + x_r}{r},$$

from the Schur-convexity result of Corollary 4 it follows that

$$\prod_{i=1}^{r} \frac{\Gamma^2(x_i+1)}{\Gamma(2x_i+1)} < \left(\frac{\Gamma^2(\bar{x}+1)}{\Gamma(2\bar{x}+1)}\right)^r$$

for $r \geq 2$ and if not all x_i are equal. Hence, a sufficient condition for (13) to hold is

$$\left(\frac{\Gamma^2(\bar{x}+1)}{\Gamma(2\bar{x}+1)}\right)^r \leq \frac{1}{2^{r\bar{x}}},$$

which, according to Corollary 1, holds if and only if $\bar{x} = 0$ or $\bar{x} \geq 1$. Therefore, the inequality (13) holds for nonnegative real numbers if $x/r = 0$ (i.e. $x_i = 0$ for all i) or $x/r \geq 1$ and the reverse inequality holds if $0 < x_i < 1$ for all i. In the intermediate case, i.e. when $0 < x/r < 1$, but with some $x_i > 1$, a computation shows that neither (13) nor the reverse inequality is generally valid.

In connection with these results, let us note that if $x_i = m_i$ are integers, then the inequality (11) yields

$$\prod_{i=1}^{n} \frac{(2m_i)!}{(m_i!)^2} \geq 2^s, \qquad s = m_1 + \cdots + m_n,$$

which is sharper than Khintchine's result in [**5**] or [**10**, p. 194]:

$$\prod_{i=1}^{n} \frac{(2m_i)!}{m_i!} \geq 2^s.$$

However, both results for integers follow from the inequality

$$\frac{(2m)!}{(m!)^2} \geq 2^m,$$

which can be proved by elementary means.

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(Received September 2, 1997)