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CONVOLUTIONS OF LOGARITHMICALLY CONCAVE FUNCTIONS

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Abstract. We give a simple proof of the fact that the logarithmic concavity of real functions is closed under convolutions. We also propose a conjecture regarding the domain of log-concavity of a convolution of two functions that are log-concave on intervals.

0. INTRODUCTION

In this paper we consider a real valued function f to be logarithmically concave (log-concave) on some interval I if $f(x) \geq 0$ and $x \mapsto \log f(x)$ is an extended real valued concave function (with a convention that $\log 0 = -\infty$). A sufficient condition for a non-negative function f to be log-concave on I is the following [**2**, 16B.3.a]:

(1) $$f(x+h)f(y) \leq f(x)f(y+h), \qquad x<y,\ x,y,x+h,y+h \in I,\ h \geq 0.$$

Log-concavity is, in a sense, more challenging property than log-convexity. For instance, a sum of two log-convex functions is again log-convex, whereas it need not hold for log-concavity.

Lekkerkerker [**5**] first discovered that a convolution of log-concave functions is log-concave. His proof of this fact was very long and elaborate; moreover, his result is restricted to functions defined on $[0, +\infty)$, under some additional requirements. Lacković in [**4**, Section 39] gave a simpler, but still long proof of Lekkerkerker's result. In this paper we present a very short and simple proof of the general result, using a natural connection between log-concavity and total positivity.

1. TOTAL POSITIVITY

Let A and B be subsets of real line. A function K defined on $A \times B$ is said to be totally positive of order k, denoted TP_k, if for all m, $1 \leq m \leq k$ and all

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$x_1 < x_2 < \cdots < x_m, y_1 < y_2 < \cdots < y_m$ $(x_i \in A, y_j \in B)$:

$$\begin{vmatrix} K(x_1, y_1) & \dots & K(x_1, y_m) \\ \vdots & & \vdots \\ K(x_m, y_1) & \dots & K(x_m, y_m) \end{vmatrix} \geq 0.$$

From the definition it immediatelly follows that if u and v are nonnegative functions and K is TP_k, then $u(x)v(y)K(x,y)$ is also TP_k.

A particular case $K(x,y) = f(y-x)$ is of a special importance. Firstly, note that K in this form is TP_2 on $A \times B$ if and only if f is non-negative and

$$f(y_1 - x_1)f(y_2 - x_2) - f(y_2 - x_1)f(y_1 - x_2) \geq 0 \tag{2}$$

for all $x_1 < x_2$ in A and all $y_1 < y_2$ in B. This condition is equivalent to log-concavity of $t \mapsto f(t)$. To see that, assume for example that $x_1 < x_2 < y_1 < y_2$ and let $y_1 - x_2 = u, y_1 - x_1 = v, y_2 - y_1 = h$. Then $u < v$ and $h > 0$ and (2) becomes

$$f(u+h)f(v) \geq f(u)f(v+h),$$

which is the condition (1) for log-concavity. This observation yields a result of Schoenberg [**1**] or [**2**, 18.A.10]: The function $K(x,y) = f(y-x)$ is TP_2 on $\mathbf{R} \times \mathbf{R}$ if and only if f is log-concave on $\mathbf{R}$.

The next result [**2**, 18.A.4.a] is fundamental in establishing the total positivity of certain functions: If K is TP_m and L is TP_n and σ is a sigma-finite measure, then the convolution

$$M(x,y) = \int K(x,z)L(z,y)\mathrm{d}\sigma(z) \tag{3}$$

is $TP_{\min(m,n)}$, see also [**2**, 18.B.1].

2. CONVOLUTIONS OF LOG-CONCAVE FUNCTIONS

In [**1**] it has been proved that any log-concave function f on $\mathbf{R}$ is either monotonic or, if it is non-monotonic then $f(x) \to 0$ as $x \to \pm\infty$ with at least an exponential rate. Therefore, if f and g are non-monotonic and log-concave functions on $\mathbf{R}$, their convolution is well defined on $\mathbf{R}$.

Theorem 1. *Let f and g be log-concave functions on $\mathbf{R}$, such that the convolution*

$$h(x) = \int_{-\infty}^{+\infty} f(x-z)g(z)\mathrm{d}z$$

is defined for all $x \in \mathbf{R}$. Then the function h is also log-concave on $\mathbf{R}$.

Proof. Let $K(x,z) = f(x-z)$ and $L(z,y) = g(z-y)$. Since f and g are log-concave functions, it follows by results of Section 1 that K and L are TP_2 on $\mathbf{R} \times \mathbf{R}$, and that

$$M(x,y) = \int_{-\infty}^{+\infty} f(x-z)g(z-y)\mathrm{d}z$$

is TP_2 on $\mathbf{R} \times \mathbf{R}$. Now note that

$$M(x,y) = \int_{-\infty}^{+\infty} f(x-y-t)g(t)\mathrm{d}t = h(x-y)$$

and since M is TP_2, it follows that h is log-concave on $\mathbf{R}$. □

Note that dz in Theorem 1 can not be replaced by arbitrary sigma-finite measure dσ, because of change of variable $t = z - y$.

Note also that if the function f is log-concave on an interval I, then the function f^* defined by

$$f^*(x) = f(x) \quad (x \in I), \qquad f^*(x) = 0 \quad \text{otherwise}$$

is log-concave on $\mathbf{R}$. Therefore, Theorem 1 is applicable to convolutions of functions defined on intervals of $\mathbf{R}$.

The statement of Theorem 1 holds if "log-concave" is replaced by "log-convex". The proof in this case is direct, by noticing that the function $x \mapsto f(x-z)g(z)$ is log-convex in z for each x.

3. LOGARITHMIC CONCAVITY IN PROBABILITY THEORY

Many frequently encountered probability densities and distribution functions are log-concave. Closely related are so called Pólya frequency functions (see [**1**] for a number of interesting properties of these functions). In [**3**] we presented some sufficient conditions for a probability distribution function to be log-concave.

By the simplest version of the Central Limit Theorem, the convolution of n identical densities converges to the Gaussian density as $n \to +\infty$. Since the Gaussian density is log-concave, this implies that under some regularity conditions, a convolution of a sufficient number of (not necessarily log-concave) densities eventually becomes a log-concave one. For a quantitative study, we need a measure of log-concavity that increases (or at least does not decrease) with convolution. Definitely, the negative second logarithmic derivative can not serve as such a measure, since it does not necessarily increase with convolution. In the next examples we present some evidence that, in fact, the convolution enlarges the domain of log-concavity.

Example 1. Let

$$f(x) = \frac{1}{x^2 + a^2} \qquad -\infty < x < +\infty, \quad a > 0.$$

Then f is log-concave on $(-a, a)$. The n-th convolution power is

$$f^{*n}(x) = \frac{n\pi^{n-1}}{a^{n-1}(x^2 + n^2a^2)}$$

and it is log-concave on $(-na, na)$.

In a special case of Cauchy density, we have

$$f(x)=\frac{1}{\pi(1+x^2)},\qquad f^{*n}=\frac{n}{\pi(x^2+n^2)}$$

and so, f is log-concave on $(-1,1)$ and its n-th convolution power is log-concave on $(-n,n)$.

In a more general case, let

$$f(x)=\frac{1}{ax^2+bx+c}\qquad -\infty<x<+\infty,\quad a>0,\ b^2-4ac<0. \tag{4}$$

Then f is log-concave on an interval I and f^{*n} is log-concave on nI.

Moreover, if the functions f and g are in the form (4) and if they are log-concave on intervals I_f and I_g respectively, then their convolution $f*g$ is log-concave on I_f+I_g.

Example 2. Doubling the domain of log-convexity with convolution is not a general rule. For example, let

$$f(x)=\frac{1}{x^4+x^2+1},\qquad f^{*2}(x)=\frac{2\sqrt{3}\pi(x^2+16)}{3(x^2+3)(x^4+4x^2+16)}.$$

Then f is log-concave on $(-1.2,1.2)$ (approximately) and f^{*2} is log-concave on $(-2.14,2.14)$. So, convolution in general does not result in doubling the interval of log-convexity, but in enlarging it.

Example 3. Although f is not log-concave on any interval where is positive, f^{*2} may be such. Let

$$f(x)=\frac{1}{x^3}\cdot H(x-1/2),$$

where H is the Heaviside step function. Then f is log-convex on $x>1/2$, but f^{*2} is log-concave on $(1,1.66)$ (approximately).

Example 4. In the previous examples we had unimodal functions. Let us now consider

$$f(x)=\sin^2(x)\quad (0\le x\le 2\pi),\qquad f(x)=0\quad \text{elsewhere}.$$

This function is log-concave on intervals $(0,\pi)$ and $(\pi,2\pi)$. Its convolution square is

$$f^{*2}(x)=\begin{cases}\dfrac{x\cos 2x}{8}-\dfrac{3\sin 2x}{16}+\dfrac{x}{4} & (0\le x\le \pi),\\[2ex] \dfrac{(4\pi-x)\cos 2x}{8}+\dfrac{3\sin 2x}{16}+\dfrac{4\pi-x}{4} & (2\pi\le x\le 4\pi)\end{cases}$$

and it is log-concave on three intervals, the first one being $(0,a)$, where $a\in(4.03,4.04)$ (numerically); hence it is longer than original interval $(0,\pi)$.

A similar situation is observed with the double triangle function.

Example 5. Let f be a given function and let $g(x)$ be constant (say, $1/2a$) on $(-a, a)$ and zero otherwise. Then $f * g$ evaluated at some point x is the mean value of f in the interval of length $2a$ centered arround x:

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(t)g(x-t)\mathrm{d}t = \frac{1}{2a}\int_{x-a}^{x+a} f(t)\mathrm{d}t.$$

If $f(x) = 1/(x^2+b^2)$, then f is log-concave on $[-b, b]$. Numerical experiments show that $f * g$ is log-concave on a longer interval; the length of that interval is positively correlated with a. □

The above examples lead to the following conjecture:

Conjecture. Let f and g be log-concave and positive functions on intervals I_f and I_g respectively. Then $f * g$ is a log-concave function on an interval I, where $\lambda(I) \geq \max(\lambda(I_f), \lambda(I_g))$ and λ is the LEBESGUE measure.

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