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On Generalized Vector Equilibrium Problem with Bounds

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We prove the existence of a solution to the generalized vector equilibrium problem with bounds. We show that several known theorems from the literature can be considered as particular cases of our results, and we provide examples of applications related to best approximations in normed spaces and variational inequalities.

1. INTRODUCTION AND PRELIMINARIES

The scalar equilibrium problem with lower and upper bounds is to find

$$x_0 \in K \text{ such that } c_1 \leq f(x_0, y) \leq c_2, \text{ for all } y \in K, \tag{1}$$

where K is a given set, $f : K \times K \to \mathbb{R}$ is a given function and c_1 and c_2 two real numbers such that $c_1 \leq c_2$. If K is a nonempty closed subset in a locally convex semi-reflexive topological vector space, then the problem (1) is problem by Isac, Sehgal and Singh [16] (Open Problem 2). In [20], Li gave the answer of this problem by introducing and the using the concept of extremal subsets. In [8], Chadli, Chang and Yao derived some results in answering this problem by using a fixed point theorem due to Ansari and Yao [4] and the Ky Fan Lemma [10]. Al-Homidan and Ansari in [14] consider system of quasi-equilibrium problems with lower and upper bounds and establish the existence of their solution by using maximal elements theorems. Liya Fan studies the existence of solution of weighted quasi-equilibrium problems with lower and upper bounds by using maximal element theorems, a fixed point theorem of set-valued maps and the Fan-KKM theorem in [13].

In this paper we study generalized vector equilibrium problem with bounds and establish some existence theorems in Hausdorff topological vector spaces. Our results improve some recent results in the literature.

Note that if $c_1 = 0$, $c_2 = 1$ and $f(x, y) = e^{-h(x,y)}$, where $h : K \times K \to \mathbb{R}$, the problem (1) is known as the scalar equilibrium problem

$$\text{find } x_0 \in K \text{ such that } h(x_0, y) \geq 0, \text{ for all } y \in K, \tag{2}$$

where K is a given set and $h : K \times K \to \mathbb{R}$ is a given function such that $h(x, x) \geq 0$ for all $x \in K$, which is a unified model of several problems, see for example [1], [2], [3], [6], [17], [21], [22], [23], [26], and reference therein.

Let X and Y be two Hausdorff topological vector spaces, K a nonempty, closed and convex subset of X, $F : K \times K \to 2^Y \setminus \{\emptyset\}$, where 2^Y is the family of all subsets of Y. The generalized vector equilibrium problem with bounds is

$$\text{find } x_0 \in K \text{ such that } F(x_0, y) \subseteq W(x_0) \text{ for all } y \in K, \tag{3}$$

where $\{W(x) : x \in K\}$ is a family of nonempty subsets of Y. Let $S_{(F,W)}$ denote the solution set of the problem (3), that is,

$$S_{(F,W)} = \{x \in K : F(x,y) \subseteq W(x) \text{ for all } y \in K\}.$$

Remark 1.1. *Suppose that $Y = \mathbb{R}$ and $F(x,y) = \{f(x,y)\}$ for all $x, y \in K$.*

- (i)**:** *If $W(x) = [c_1, c_2]$ for all $x \in K$, then the problem (3) reduces to problem (1).*
- (ii)**:** *If $W(x) = [0, +\infty)$, for all $x \in K$, then the problem (3) reduces to the problem (2).*
- (iii)**:** *If $\{W(x) : x \in K\}$ is a family of convex cones such that $W(x) \neq Y$ and $intW(x) \neq \emptyset$ for all $x \in K$ where $intW(x)$ denotes the interior of $W(x)$, then the problem (3) reduces to the generalized vector equilibrium problem:*

$$\textit{find } x_0 \in K \textit{ such that } f(x_0, y) \in W(x_0) \textit{ for all } y \in K.$$

For a nonempty subset D of X, let $\langle D\rangle$ denote the set of all nonempty finite subsets of D. Let K be a convex subset of X, a map $H : K \to 2^X$ is called KKM map if $coA \subseteq \bigcup_{a\in A} H(a)$ for each $A \in \langle K\rangle$.

In this paper we will use Fan's generalization [12] of the KKM theorem which is also a generalization of [10] (Lemma 1). We state here a slightly modified form of that result (see Lemma 1 and Theorem 4 in [12]).

Theorem 1.1. *Fan-KKM Theorem. Let X be a Hausdorff topological vector space, K a nonempty subset of X and $H : K \to 2^X$ a KKM map with closed values. Let D a nonempty subset of K, which is contained in a compact convex subset of X. Then*

$$S := \bigcap_{x\in D} H(x) \neq \emptyset.$$

In addition, if the set S is compact, then

$$\bigcap_{x\in K} H(x) \neq \emptyset.$$

2. Main Results

Theorem 2.1. *Let X and Y be Hausdorff topological vector spaces, let K be a nonempty convex subset of X and let $\{W(x) : x \in K\}$ be a family of nonempty subsets of Y. Assume that map $F : K \times K \to 2^Y$ satisfies the following conditions:*

1. *the set $\{x \in K : F(x,y) \subseteq W(x)\}$ is closed for all $y \in K$,*
2. *$co\{y : F(x,y) \nsubseteq W(x)\} \subseteq \{y : F(x,y) \neq F(x,x)\}$ for all $x \in K$,*
3. *there exists a nonempty subset D of K which is contained in a compact and convex subset of X such that*

$$\bigcap_{y\in D} \{x \in K : F(x,y) \subseteq W(x)\}$$

is a compact subset of K.

Then the set $S_{(F,W)}$ is nonempty and compact.

Proof. Define $H : K \to 2^K$ by

$$H(y) = \{x \in K : F(x,y) \subseteq W(x)\}.$$

From condition (2) we obtain

$$F(y,y) \subseteq W(y) \text{ for all } y \in K,$$

so $H(y)$ is a nonempty set for all $y \in K$, because $y \in H(y)$.

Note that by condition (1), the set $H(y)$ is closed for each $y \in K$. The map $y \mapsto H(y)$ is a KKM map. Indeed, suppose that for some $D \in \langle K \rangle$,

$$coD \nsubseteq \bigcup_{d\in D} H(d).$$

Then there exists $y \in coD$ such that $y \notin H(d)$ for every $d \in D$. So, we have

$$F(y,d) \nsubseteq W(y) \text{ for every } d \in D.$$

From condition (2) we obtain

$$F(y,y) \neq F(y,y).$$

This is a contradiction and H is a KKM map. Now, from Theorem 1.1 it follows that there exists $x_0 \in K$ such that

$$F(x_0,y) \subseteq W(x_0) \text{ for all } y \in K.$$

So, the set $S_{(F,W)}$ is nonempty. $S_{(F,W)}$ is compact by condition (1) and (3). This completes the proof. $\square$

The following theorem follows immediately from Theorem 2.1 and first part of the Theorem 1.1.

Theorem 2.2. *Let X and Y be Hausdorff topological vector spaces, let K be a nonempty convex set which is contained in a compact subset of X and let $\{W(x) : x \in K\}$ be a family of nonempty subsets of Y. Assume that map $F : K \times K \to 2^Y$ satisfies the following conditions*

1. *the set $\{x \in K : F(x,y) \subseteq W(x)\}$ is closed for all $y \in K$,*
2. *$co\{y : F(x,y) \nsubseteq W(x)\} \subseteq \{y : F(x,y) \neq F(x,x)\}$ for all $x \in K$,*

Then the set $S_{(F,W)}$ is nonempty and compact.

Note that, if $F(x,x) \subseteq W(x)$ for all $x \in K$, then we have

$$\{y : F(x,y) \nsubseteq W(x)\} \subseteq \{y : F(x,y) \neq F(x,x)\}$$

for all $x \in K$. Now, if the set $\{y : F(x,y) \nsubseteq W(x)\}$ is convex for all $x \in K$ then condition (2) of Theorem 2.1 is satisfied. So, from Theorem 2.1 we obtain following result.

Theorem 2.3. *Assume that a map $F : K \times K \to 2^Y$ satisfies the following conditions:*

1. *$F(x,x) \subseteq W(x)$ for all $x \in K$,*
2. *the set $\{x \in K : F(x,y) \subseteq W(x)\}$ is closed for all $y \in K$,*
3. *the set $\{y : F(x,y) \nsubseteq W(x)\}$ is convex for all $x \in K$,*
4. *there exists a nonempty subset D of K which is contained in a compact convex subset of X such that*

$$\bigcap_{y\in D} \{x \in K : F(x,y) \subseteq W(x)\}$$

is a compact subset of K.

Then the set $S_{(F,W)}$ is nonempty and compact.

Remark 2.1. (1) *Let* $W(x) = Y \setminus (-intC(x))$ *for all* $x \in K$, *where* $\{C(x) : x \in K\}$ *is a family of convex cones with* $intC(x) \neq \emptyset$ *for all* $x \in K$ *and* $F(x, \cdot)$ *is* $C(x)-$*convex, i. e. for all* $y_1, y_2 \in K$ *and* $\lambda \in [0, 1]$,

$$\lambda F(x, y_1) + (1 - \lambda)F(x, y_2) \subseteq F(x, \lambda y_1 + (1 - \lambda)y_2) + C(x),$$

for all $x \in K$, *then condition* (3) *of Theorem 2.3 holds. So, from Theorem 2.3 we obtain Theorem 2.1 of A. H. Wan, J. Y. Fu and W. H. Mao* [27].

(2) *From Theorem 2.3 we obtain Theorem 2.4 of A. P. Farajzadeh* [9].

Example 2.1. *Let* $X = \mathbb{R}$, $Y = \mathbb{R}$, $K = [0, 1]$, $W(x) = [0, +\infty)$ *for all* $x \in [0, 1]$ *and define a set-valued map* $F : K \times K \to 2^{\mathbb{R}}$ *with*

$$F(x, y) = \begin{cases} \{-1\}, & x \in (0, 1), y \in [0, x^3) \cup (x^2, x), \\ \{1\}, & x \in (0, 1), y \in [x^3, x^2], \\ \{1\}, & x \in \{0, 1\}, y \in [0, 1], \\ \{2\}, & x \in (0, 1), y \in [x, 1], \end{cases}$$

Then

$$\{x \in K : F(x, y) \subseteq W(x)\} = [0, y] \cup [\sqrt{y}, \sqrt[3]{y}] \cup \{1\} \text{ for all } y \in K,$$

hence $\{x \in K : F(x, y) \subseteq W(x)\}$ *is closed in* K *for all* $y \in K$. *For sets* $\{y : F(x, y) \nsubseteq W(x)\}$ *and* $\{y : F(x, y) \neq F(x, x)\}$, *we obtain*

$$\{y : F(x, y) \nsubseteq W(x)\} = [0, x^3) \cup (x^2, x) \text{ if } x \in (0, 1),$$
$$\{y : F(x, y) \nsubseteq W(x)\} = \emptyset \text{ if } x \in \{0, 1\},$$
$$F(x, x) = \{2\}, \text{ if } x \in (0, 1),$$
$$F(x, x) = \{1\}, \text{ if } x \in \{0, 1\},$$
$$\{y : F(x, y) \neq F(x, x)\} = (0, x) \text{ if } x \in (0, 1),$$
$$\{y : F(x, y) \neq F(x, x)\} = \emptyset \text{ if } x \in \{0, 1\},$$

and F *satisfies all hypotheses in Theorem 2.1. In this case we have* $S_{(F,W)} = \{0, 1\}$. *Note that set* $\{y : F(x, y) \nsubseteq W(x)\}$ *is not convex if* $x \in (0, 1)$ *and hence Theorem 2. 4 in* [9] *is not applicable.*

3. Some Applications

3.1. **Applications to equilibrium theory.** Here we single out three corollaries to Theorems 2.1 and 2.2 related to the existence of solutions for vector equilibrium problems and scalar equilibrium problems with lower and upper bounds.

Corollary 3.1. *Let* X *and* Y *be Hausdorff topological vector spaces, let* K *be a nonempty convex compact subset of* X *and let* $\{W(x) : x \in K\}$ *a family of nonempty subsets of* Y. *Assume that a vector-valued map* $f : K \times K \to Y$ *satisfies the following conditions*

(1) *the set* $\{x \in K : f(x, y) \in W(x)\}$ *is closed in* K *for all* $y \in K$,
(2) $co\{y : f(x, y) \notin W(x)\} \subseteq \{y : f(x, y) \neq f(x, x)\}$ *for all* $x \in K$,

then there exists $x_0 \in K$, *such that*

$$f(x_0, y) \in W(x_0) \text{ for all } y \in K.$$

Proof. The result follows directly from Theorem 2.2 with

$$F(x, y) = \{f(x, y)\} \text{ for all } x, y \in K.$$

□

Remark 3.1. (1) *Let* $W(x) = Y \setminus intC$ *for all* $x \in K$*, where* C *is a closed convex and pointed cone with* $intC \neq \emptyset$*. In this case from Corollary 3.1 we obtain Theorem 1.1 of H. Yang and J. Yu* [25].

(2) *If we put* $W(x) = -intC$ *for all* $x \in K$ *and* $F(x,y) = \{f(g(x),y)\}$ *for all* $x, y \in K$ *from Corollary 3.1, we obtain Theorem 4.2 of N. J. Huang and H.B. Thompson* [15].

Corollary 3.2. *Let* X *be a Hausdorff topological vector space,* K *a nonempty convex compact subset of* X *and let* f *be a real bifunction on* $K \times K$*. If there exist two real functions* g *and* h *on* K *such that*

1. *the set* $\{x \in K : g(x) \leq f(x,y) \leq h(x)\}$ *is closed in* K *for all* $y \in K$*,*
2. $co\{y : f(x,y) \in \mathbb{R} \setminus [g(x), h(x)]\} \subseteq \{y : f(x,y) \neq f(x,x)\}$ *for all* $x \in K$*,*

then there exists $x_0 \in K$*, such that*

$$g(x_0) \leq f(x_0, y) \leq h(x_0) \textit{ for all } y \in K.$$

Proof. The result follows directly from Theorem 2.2 with

$$F(x,y) = \{f(x,y)\} \text{ for all } x, y \in K,$$

$$W(x) = [g(x), h(x)] \text{ for all } x \in K.$$

□

Corollary 3.3. *Let* X *be a Hausdorff topological vector space,* K *a nonempty convex closed subset of* X *and let* f *be a real bifunction on* $K \times K$*. If there exists real function* g *on* K *such that*

1. *the set* $\{x \in K : f(x,y) \geq g(x)\}$ *is closed for all* $y \in K$*,*
2. $co\{y : f(x,y) < g(x)\} \subseteq \{y : f(x,y) \neq f(x,x)\}$ *for all* $x \in K$*,*
3. *there exists a nonempty subset* D *of* K *which is contained in a compact convex subset of* X *such that*

$$\bigcap_{y \in D} \{x \in K : f(x,y) \geq g(x)\}$$

is a compact subset of K*,*

then there exists $x_0 \in K$*, such that*

$$f(x_0, y) \geq g(x_0) \textit{ for all } y \in K.$$

Proof. The result follows directly from Theorem 2.1 with

$$F(x,y) = \{f(x,y)\} \text{ for all } x, y \in K,$$

$$W(x) = [g(x), +\infty) \text{ for all } x \in K.$$

□

Remark 3.2. (1) *Let* c_1 *and* c_2 *be two real numbers such that* $c_1 \leq c_2$ *and put* $g(x) = c_1$ *and* $h(x) = c_2$ *for all* $x \in K$*. In this case Corollary 3.2 yields Theorem 3.1 in* [20].

(2) *If* $g(x) = c$ *for all* $x \in K$*, where* c *is a real number, then from Corollary 3.3 follows Theorem 3 in* [19].

3.2. Applications on best approximations.

(1) (Ky Fan [11], Best approximation theorem.) Let K be a nonempty compact, convex subset of a normed linear space X and $f : K \to X$ a continuous function. Then there is an $x_0 \in K$ such that

$$||x_0 - f(x_0)|| = \inf_{x \in K} ||x - f(x_0)||.$$

Put $g(x) = ||x - f(x)||, f(x, y) = ||y - f(x)||$ for $x, y \in X$, in Corollary 3.3.

(2) (Prolla [24], Best approximation theorem.) Let K be a nonempty compact, convex subset of a normed linear space X and $f : K \to X$ a continuous function and $g : K \to X$ a continuous, almost-affine, onto map. Then there is an $x_0 \in K$ such that

$$||g(x_0) - f(x_0)|| = \inf_{x \in K} ||x - f(x_0)||.$$

Put $g(x) = ||g(x) - f(x)||, f(x, y) = ||g(y) - f(x)||$ for $x, y \in X$, in Corollary 3.3.

3.3. Applications on variational inequalities.

(1) (Browder, [7] Theorem 3.) Let E be a topological vector space on which its topological dual E^* is equipped with a topology such that the pairing $\langle , \rangle : E^* \times E \to \mathbb{R}$ is continuous. Let K be a compact convex subset of E, and $T : K \to E^*$ continuous. Then there exists a $x_0 \in K$ such that

$$\langle T(x_0), y - x_0 \rangle \geq 0 \text{ for all } y \in K.$$

Put $g(x) = 0, f(x, y) = \langle T(x), y - x \rangle$ for $x, y \in X$, and apply Corollary 3.3.

(2) (Karamardian, [18] Lemma 3.2) Let X be a compact convex subset of a topological vector space E, F a topological space, $g : X \to F$ a function, and $\psi : X \times F \to \mathbb{R}$ a function. If for each $y \in F, \psi(, y)$ is quasiconvex on X and the function $(u, v) \to \psi(u, g(v))$ is continuous on $X \times X$, then there exists an $x_0 \in X$ such that

$$\psi(x_0, g(x_0)) \leq \psi(y, g(x_0)) \text{ for all } y \in X.$$

Put $g(x) = \psi(x, g(x)), f(x, y) = \psi(y, g(x))$ for $x, y \in X$, and apply Corollary 3.3.

(3) (Behera and Panda, [5] Theorem 2. 2) Let X be a compact convex subset of a topological vector space E on which E^* is equipped with a topology such that the pairing $\langle , \rangle : E^* \times E \to \mathbb{R}$ is continuous, $T : X \to E^*$ and $\theta : K \times K \to E$ continuous maps such that
(i) $\langle T(y), \theta(y, y) \rangle \geq 0$ for all $y \in X$,
(ii) for each $y \in X$, the function $\langle T(y), \theta(\cdot, y) \rangle : X \to \mathbb{R}$ is quasiconvex.
Then there exists an $x_0 \in X$ such that

$$\langle T(x_0), \theta(y, x_0) \rangle \geq 0 \text{ for all } y \in X.$$

Put $g(x) = 0, f(x, y) = \langle T(x), \theta(y, x) \rangle$ for $x, y \in X$, and apply Corollary 3.3.

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