Acta Math. Hungar.
80 (1–2) (1998), 1–7.

COMPLETION OF COUNTABLY SEMINORMED SPACES

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Abstract. Let E be a vector space with a topology generated by countably many increasing seminorms $p_1 \leqq p_2 \leqq \dots$ and let $\bar{E}_{p_i}$ be the completion of $E/\ker(p_i)$ with respect to p_i. If p_i are norms compatible in a certain sense, it is known that E is a complete space if and only if $E = \cap_{i=1}^{\infty} \bar{E}_{p_i}$. In this paper we give a similar characterization of complete spaces in a general case when p_i are seminorms, without any additional assumptions. Our characterization coincides with the known one if p_i are compatible norms.

1. Introduction and definitions

Let E be a real vector space. A real valued function p defined on E is called a seminorm if for all $x, x_1, x_2 \in E$ and all $a \in R$: $p(x) \geqq 0$, $p(ax) = |a|p(x)$, $p(x_1 + x_2) \leqq p(x_1) + p(x_2)$ and $p(x) > 0$ for some $x \in E$.

Let $E_p = (E, p)$, i.e. E with the topology defined by a seminorm p. Let $\ker(p) = \{x \in E \mid p(x) = 0\}$. Then $E_p/\ker(p)$ is a normed space. Let $\bar{E}_p$ be the completion of $E/\ker(p)$ with respect to p.

If $p < q$ then the identity mapping $i_{p,q} : E_q \to E_p$ is continuous. It can be extended to $i_{p,q} : E_q/\ker(q) \to E_p/\ker(p)$ by

$$i_{p,q}\big(x + \ker(q)\big) = x + \ker(p). \tag{1}$$

The map (1) is well defined because, by $p < q$, the implication $y_1, y_2 \in x + \ker(q) \Rightarrow y_1, y_2 \in x + \ker(p)$ holds. However, it is not injective, as the reverse implication does not hold.

For $x \in \bar{E}_q$ there is a sequence $x_n \in E_q/\ker(q)$, $q(x_n - x) \to 0$. Since $\{x_n\}$ is a q-Cauchy sequence, $i_{p,q}(x_n)$ is a p-Cauchy sequence. Then there is a $y \in \bar{E}_p$, such that $p\text{-}\lim i_{p,q}(x_n) = y$. Define $i_{p,q}(x) = y$. The mapping $i_{p,q}$ is now extended to $i_{p,q} : \bar{E}_q \to \bar{E}_p$. This is a linear continuous mapping, which also may not be injective in general.

For $x \in \bar{E}_q$, let us define $p(x) = p(i_{p,q}x)$. On $\bar{E}_q$, p is only a seminorm.

Let Π be a countable family of seminorms, $\Pi = \{p_i\}$, such that $p_1(x) \leqq p_2(x) \leqq \dots$ and suppose that for every $x, y \in E$, $x \neq y$, there is a seminorm

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[1] Research supported by Science Fund of Serbia, grant number 0401A, through Mathematical Institute, Belgrade.

0236–5294/98/$ 5.00

$p \in \Pi$ such that $p(x-y) \neq 0$. For $x_1, x_2 \in E$, define

$$d(x_1,x_2) = \sum_{i=1}^{\infty} 2^{-i} \frac{p_i(x_1-x_2)}{1+p_i(x_1-x_2)}. \tag{2}$$

Then d is a metric on E; the space E with the topology determined by d is called a countably seminormed space, denoted by (E,Π,d). Equivalently, the topology may be defined in terms of neighborhood basis at 0, which consists of sets $\left\{x \in E \,|\, p_{i_1}(x) < \varepsilon_1, \dots, p_{i_k}(x) < \varepsilon_k\right\}$.

Note that $d(x_n,x) \to 0$ if and only if $p(x_n - x) \to 0$ for all $p \in \Pi$ and that a sequence $\{x_n\}$ is a Cauchy sequence in the metric d if and only if it is a Cauchy sequence in each $p \in \Pi$.

If p_i are norms, the space (E,Π,d) is called a countably normed space [1, 2, 6, 7]. A case of Hilbertian seminorms has been studied in connection with some stochastic differential equations [3, 4, 5, 8, 9].

If d is defined by (2), then $x \mapsto d(x,0)$ is a quasinorm and by a completion theorem [10, I.10], (E,Π,d) can be densely embedded into a complete space. In case of norms, under some additional requirements, a necessary and sufficient condition for completeness of (E,Π,d) is known (Theorem 1 in Section 2). In Sections 3 and 4 we propose a way of completion which yields a similar condition in a general case, without additional assumptions.

2. Completeness in case of compatible norms

If p_i are norms, then the maps $i_{p,q} : E_q/\ker(q) \to E_p/\ker(p)$ are just plain identity maps. To make maps $i_{p,q} : \bar{E}_q \to \bar{E}_p$ injective, a condition of compatibility of norms is proposed in [1] and [2, 1.2.2].

If p, q $(p<q)$ are norms on E, we say that they are compatible if any q-Cauchy sequence $\{x_n\}$ that converges to 0 in p-norm, also converges to 0 in q-norm.

If the topology of (E,Π,d) is defined by compatible norms, then $\bar{E}_q \subset \bar{E}_p$ for $p<q$ and it makes sense to consider $\cap_i \bar{E}_{p_i}$. The latter expression appears in the following completion theorem [2, 1.3.2]:

THEOREM 1. *Let (E,Π,d) be a countably normed space, with $\Pi = \{p_i\}_{i=1}^{\infty}$, where for every $i<j$, p_i and p_j are compatible norms. Then E is complete with respect to the metric d if and only if*

$$E = \bigcap_{i=1}^{+\infty} \bar{E}_{p_i}. \qquad \square \tag{3}$$

However, the expression on the right hand side of (3) is meaningless if p_i are seminorms, or if p_i are norms, but not compatible.

3. Seminorm completion

Let (E, Π, d) be a countably seminormed space. We define a seminorm completion $\bar{E}_p^s$ as follows.

Let $\bar{E}_p^s$ be the space of all p-Cauchy sequences $\{x_n\}$ on E with a seminorm

$$p(\{x_n\}) = \lim_{n\to\infty} p(x_n).$$

The limit here exists, because $p(x_n)$ is a Cauchy sequence in R. Then $(\bar{E}_p^s, p)$ is a complete seminormed space, which can be shown by a standard argument. Note that we do not take equivalence classes as elements of $\bar{E}_p^s$; this is why we obtain only a seminormed space, even if p is a norm.

Define a map $\pi : E \to \bar{E}_p^s$ by $\pi(x) = \{x, x, \dots, x, \dots\}$. This makes a 1–1 isometrical and isomorphical mapping of E_p onto a dense linear subspace of the space $\bar{E}_p^s$.

A semi-metric d on $\bar{E}_p^s$ is defined by

$$d(\{x_n\}, \{y_n\}) = \sum_{i=1}^{\infty} \limsup_{n\to\infty} \frac{p_i(x_n - y_n)}{1 + p_i(x_n - y_n)}. \tag{4}$$

Then $d(\{x_n\}, \{y_n\}) = 0$ if and only if for all $p_i \in \Pi$, $\lim_{n\to\infty} p_i(x_n - y_n) = 0$. So, an equivalence relation can be defined in $\bar{E}_p^s$ by

$$\{x_n\} \sim \{y_n\} \iff d(\{x_n\}, \{y_n\}) = 0.$$

The set of all such equivalence classes, equipped with the topology of $\bar{E}_p^s$ will be denoted by $\bar{E}_p^s / \ker(d)$.

It is easy to see that $\bar{E}_{p_i}^s / \ker(d) \supset \bar{E}_{p_j}^s / \ker(d)$ for $i < j$, and we can define

$$\bar{E}^s = \bigcap_{i=1}^{\infty} \bar{E}_{p_i}^s / \ker(d) = \Big(\bigcap_{i=1}^{\infty} \bar{E}_{p_i}^s\Big) / \ker(d). \tag{5}$$

From the above construction it follows that d is a metric on $\bar{E}^s$. In the next section we show that $\bar{E}^s$ is a complete space with respect to d.

4. Completeness of countably seminormed spaces

THEOREM 2. *Let (E, Π, d) be a countably seminormed space. Then $\bar{E}^s$ defined by* (5) *is a complete space. Moreover, there is an isometric and isomorphic,* 1–1 *mapping π of E onto a dense linear subspace of $\bar{E}^s$.*

(E, Π, d) is a complete space if and only if

$$\pi(E) = \bar{E}^s \tag{6}$$

If (6) *holds, we may write $E = \bar{E}^s$.*

PROOF. Elements of $\bar{E}^s$ are equivalence classes of sequences that are p_i-Cauchy for all i. For any such sequence x_n, $\lim_n p_i(x_n)$ exists and is finite. So, (4) becomes

$$d(\{x_n\}, \{y_n\}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{\lim_n p_i(x_n - y_n)}{1 + \lim_n p_i(x_n - y_n)}. \tag{7}$$

If $d(\{x_n\}, \{y_n\}) = 0$, then the sequences $\{x_n\}$ and $\{y_n\}$ belong to the same equivalence class in $\bar{E}^s$. Therefore, d is a metric on $\bar{E}^s$. To show its completeness, we proceed in a standard way.

Suppose that $\bar{x}_k = \{x_{k,n}\}_k$ is a d-Cauchy sequence in E^s (more precisely, $\{x_{k,n}\}_k$ is a Cauchy sequence of equivalence classes in $\bar{E}^s$). Then for each fixed k, $\bar{x}_k$ is an equivalence class of $\bar{E}^s$ that contains the Cauchy sequence $x_{k,1}, x_{k,2}, \dots, x_{k,n}, \dots$ of E, and we have

$$d(\bar{x}_j, \bar{x}_k) = \lim_{n\to\infty} d(x_{j,n}, x_{k,n}) \to 0 \quad \text{as} \quad j, k \to \infty. \tag{8}$$

For each fixed k, choose n_k such that

$$d(x_{k,m}, x_{k,n_k}) < \frac{1}{k} \quad \text{if} \quad m \geqq n_k, \tag{9}$$

which is possible because, for each fixed k, the sequence $\{x_{k,n}\}_n$ is Cauchy in E. Define $\bar{x}$ to be the equivalence class that contains the sequence

$$x_{1,n_1}, x_{2,n_2}, \dots, x_{k,n_k}, \dots . \tag{10}$$

By (9), for $m \geqq \max(n_j, n_k)$ we have

$$\begin{aligned} d(x_{j,n_j}, x_{k,n_k}) &\leqq d(x_{j,n_j}, x_{j,m}) + d(x_{j,m}, x_{k,m}) + d(x_{k,m}, x_{k,n_k}) \\ &\leqq \frac{1}{j} + \frac{1}{k} + d(x_{j,m}, x_{k,m}). \end{aligned}$$

Letting $m \to \infty$ and then $j, k \to \infty$ we get, using (8),

$$d(x_{j,n_j}, x_{k,n_k}) \to 0 \quad \text{as} \quad j, k \to \infty,$$

and so, (10) is a Cauchy sequence in E and $\bar{x}$ is in $\bar{E}^s$.

For any fixed j, k, with $j > n_k$ we have

$$d(x_{k,j}, x_{j,n_j}) \leqq d(x_{k,j}, x_{k,n_k}) + d(x_{k,n_k}, x_{j,n_j}) \leqq \frac{1}{k} + d(x_{k,n_k}, x_{j,n_j}).$$

Letting $j, k \to +\infty$ we get $\lim\limits_{k \to +\infty} d(\bar{x}_k, \bar{x}) = 0$, which shows that $\bar{E}^s$ is a complete space.

For every $p \in \Pi$ and $\{x_n\} \in \bar{E}^s$, we may define

$$p\big(\{x_n\}\big) = \lim_{n \to \infty} p(x_n).$$

So, $d\big(\{x_n\}, \{y_n\}\big) \to 0$ if and only if $p\big(\{x_n\} - \{y_n\}\big) \to 0$ for every $p \in \Pi$. Note that p need not be a norm on $\bar{E}^s$ even if it is a norm on E.

Let us define a mapping $\pi : E \to \bar{E}^s$ by

$$\pi(x) = \{x, x, \dots, x, \dots\} + \ker(d). \tag{11}$$

Then $\pi(E)$ is a linear subspace of $\bar{E}^s$ and π is an isomorphism. From (7) it follows

$$d\big(\pi(x), \pi(y)\big) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{p_i(x-y)}{1 + p_i(x-y)} = d(x, y), \tag{12}$$

so π is an isometry. From (11) and (12) it follows that π is a 1–1 mapping.

Let us show that $\pi(E)$ is dense in E^s. Let $\{x_n\} + \ker(d) \in E^s$. Then $\{x_n\}$ is a d-Cauchy sequence in E, so $\pi(x_n)$ is a d-Cauchy sequence in $\pi(E)$, and, as $n \to \infty$ we have

$$d\big(\pi(x_n) - \{x_n\}\big) \to 0;$$

hence, $\pi(E)$ is dense in $\bar{E}^s$.

Now we show that (6) is a necessary and sufficient condition for completeness of E. If (6) holds, then E is complete, because $\bar{E}^s$ is so. Conversely, let (E, Π, d) be a complete space. Let π be defined by (11). We need to show that $\bar{E}^s \subset \pi(E)$. To this end, let $y = \{x_n\} + \ker(d) \in \bar{E}^s$. Then $\pi(x_n) \to y$ in the metric d. By completeness of E, there is a $z \in E$ such that $x_n \to z$. By the uniqueness of a limit, we have that $y = \pi(z) \in \pi(E)$, which was to be proved. $\square$

We will now show that our construction by means of seminorm completion yields the same result as the classical one in a special case of compatible norms.

THEOREM 3. *Let* (E, Π, d) *be a countably normed space, with* $\Pi = \{p_i\}_{i=1}^{\infty}$, *where for every* $i < j$, p_i *and* p_j *are compatible norms. Then*

$$\bigcap_{i=1}^{\infty} \bar{E}^s_{p_i} / \ker(d) = \bigcap_{i=1}^{\infty} \bar{E}_{p_i}. \tag{13}$$

PROOF. Elements of the set on the left hand side of (13) are equivalence classes of Cauchy sequences in E, defined in such a way that two Cauchy sequences x_n,y_n belong to the same class if and only if $d(x_n - y_n) \to 0$. By $\bar{E}^s_{p_1} \subset \bar{E}^s_{p_2} \subset \ldots$, sequences in the intersection are Cauchy in all p_i, and so they are d-Cauchy sequences.

Elements of the set on the right hand side of (13) are also equivalence classes with respect to $\ker(p_1)$, i.e, two d-Cauchy sequences belong to the same class if and only if $p_1(x_n - y_n) \to 0$.

The compatibility of norms is, indeed, equivalent to $\ker(d) = \ker(p_1)$ in the space of all Cauchy sequences; that is, for all d-Cauchy sequences $\{x_k\}$, $p_1(x_k) \to 0$ if and only if $d(x_k) \to 0$.

Therefore, (13) is proved. □

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(Received November 25, 1996)

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