Aequ. math. 54 (1997) 173–180
0001-9054/97/010173-08 \$ 1.50 + 0.20/0

Aequationes Mathematicae

Inequalities for sums of independent geometrical random variables

MILAN MERKLE AND LJILJANA PETROVIĆ

Summary. We give a survey of known results regarding Schur-convexity of probability distribution functions. Then we prove that the function $F(p_1, \ldots, p_n; t) = P(X_1 + \cdots + X_n \le t)$ is Schur-concave with respect to $(p_1, \ldots, p_n)$ for every real t, where X_i are independent geometric random variables with parameters p_i. A generalization to negative binomial random variables is also presented.

1. Introduction and known results

Let us firstly review concepts of majorization and Schur-convexity (see [6] for details). Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbf{R}^n$, and let $x_{[i]}, y_{[i]}$ denote the i-th largest component of $\mathbf{x}, \mathbf{y}$ respectively. Then we say that $\mathbf{x} \prec \mathbf{y}$ ($\mathbf{x}$ is majorized by $\mathbf{y}$) if

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, \qquad k = 1, 2, \ldots, n-1, \qquad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}.$$

A function f of n arguments is Schur-convex on a set $A \subset \mathbf{R}^n$ if, for all $\mathbf{x}, \mathbf{y} \in A$,

$$\mathbf{x} \prec \mathbf{y} \Rightarrow f(\mathbf{x}) \le f(\mathbf{y}).$$

A function f is Schur-concave if

$$\mathbf{x} \prec \mathbf{y} \Rightarrow f(\mathbf{x}) \ge f(\mathbf{y}).$$

Let f be a Schur-convex function on a convex set A. Then

AMS subject classification (1991): Primary 60E15, Secondary 62E99.

Manuscript received July 27, 1995.

$$f(x_1, x_2, \ldots, x_n) \geq f(m, m, \ldots, m), \qquad m = \frac{x_1 + \cdots + x_n}{n}. \tag{1}$$

The inequality (1) is a most prominent consequence of Schur-convexity. For a Schur-concave function f, the inequality (1) holds with $\leq$ in place of $\geq$.

Of course, other ways of applying Schur-convexity also produce useful inequalities that are sometimes very hard to prove using other methods.

In probability and statistics one commonly deals with the functions of the form

$$P(c_1 X_1 + c_2 X_2 + \cdots + c_n X_n \leq t),$$

where $X_1, \ldots, X_n$ are continuous independent random variables with the same distribution and $c_i \geq 0$. It makes sense to investigate Schur-convexity of these functions with respect to $(c_1, \ldots, c_n)$, because it leads to interesting conclusions about the behaviour of tail probabilities for the linear combination of random variables (see [1] or [2] for examples of applications).

In a discrete case, it is natural to study functions of the form

$$P(X_1 + X_2 + \cdots + X_n \leq t),$$

where $X_1, \ldots, X_n$ are discrete independent random variables from the same class of distributions, such that X_i has a certain distribution with a parameter θ_i. Here one investigates Schur-convexity with respect to $(\theta_1, \ldots, \theta_n)$.

The most general result is due to Proschan [7]: If $X_1, \ldots, X_n$ are independent absolutely continuous random variables, with a common density which is symmetric about zero and log-concave, then the function

$$F(c_1, \ldots, c_n; t) = P(c_1 X_1 + \cdots + c_n X_n \leq t), \qquad c_i \geq 0, \quad i = 1, 2, \ldots, n, \tag{2}$$

is Schur-concave in $(c_1, \ldots, c_n)$ for every $t \geq 0$.

For non-symmetric continuous distributions only several particular cases have been dealt with in detail. Most research has been done for the Gamma distribution.

Let X_i have a Gamma(α, β) density

$$f(x) = e^{-\beta x} x^{\alpha - 1} \frac{\beta^\alpha}{\Gamma(\alpha)}, \quad (x \geq 0), \qquad f(x) = 0, \quad (x < 0).$$

Let F be defined as in (2). By a result in [1], for $n = 2$, F is Schur-convex in $\mathbf{c}$ if $t \leq \alpha(c_1 + c_2)/\beta$ and is Schur-concave for $t \geq (\alpha + \frac{1}{2})(c_1 + c_2)/\beta$.

For a general n and the same Gamma distribution, it is proved in [1] that F is Schur-convex in $\mathbf{c}$ in the region

$$\left\{c\colon \min_{1 \le i \le n} c_i \ge \frac{t\beta}{n\alpha + 1}\right\},$$

and this function is Schur-concave in $\mathbf{c}$ for

$$t \ge \frac{(n\alpha + 1)(c_1 + \cdots + c_n)}{\beta}.$$

By a result of Tong [8], the function F is Schur-concave with respect to $(c_1^{-1}, \ldots, c_n^{-1})$ for all $t \ge 0$ and all $n \ge 2$.

If X_1, X_2 are independent random variables with a Weibull density

$$f(x) = \gamma\beta x^{\beta - 1} e^{-\gamma x^\beta}, \quad (x > 0), \qquad f(x) = 0 \quad (x \le 0),$$

then (according to [1]), the function F (for $n = 2$) is Schur-concave in $\mathbf{c}$ if

$$t \ge C(\beta)(c_1 + c_2)\left(\frac{1}{\gamma}\left(1 + \frac{1}{2\beta}\right)\right)^{1/\beta},$$

where $C(\beta) = 2^{(1/\beta) - 1}$ for $0 < \beta < 1$ and $C(\beta) = 1$ for $\beta \ge 1$.

For discrete distributions, there is a result of Gleser [3] for Bernoulli distribution: Let $X_1, \ldots, X_n$ be independent Bernoulli random variables with $P(X_i = 1) = p_i$, $i = 1, 2, \ldots, n$. Then the function

$$F(p_1, \ldots, p_n; t) = P(X_1 + \cdots + X_n \le t)$$

is Schur-concave in $\mathbf{p}$ for $0 \le t \le n\bar{p} - 2$ and it is Schur-convex for $n\bar{p} + 1 \le t \le n$, where $\bar{p} = (p_1 + \cdots + p_n)/n$.

A result of Kanter [5] states that if X_i, $i = 1, 2, \ldots, n$ are independent random variables with

$$P(X_i = 1) = P(X_i = -1) = \frac{\lambda_i}{2}, \qquad P(X_i = 0) = 1 - \lambda_i,$$

then $P(X_1 + \cdots + X_n = 0 \text{ or } m)$ is, for every $m > 0$ a Schur-concave function of $(\lambda_1, \ldots, \lambda_n)$.

2. Geometric and negative binomial distributions

In this part we present our results related to geometric distributions.

THEOREM 1. *Let* $X_1, \ldots, X_n$ *be independent geometric random variables with parameters* $p_1, \ldots, p_n$ *respectively*:

$$P(X_i = k) = p_i(1-p_i)^{k-1}, \qquad k = 1, 2, \ldots .$$

Let $\mathbf{p} = (p_1, \ldots, p_n)$. *The function*

$$F(p_1, \ldots, p_n; t) = P(X_1 + \cdots + X_n \leq t)$$

is Schur-concave with respect to $\mathbf{p}$ *for every real* t.

Proof. By [6], it suffices to show that the function

$$x \mapsto F(p_1, c - p_1, p_3, \ldots, p_n, t)$$

is decreasing in p_1 when $p_1 \in (c/2, c)$, with $p_3, \ldots, p_n$ being fixed. Since t is arbitrary and

$$P(X_1 + \cdots + X_n \leq t) = E(P(X_1 + X_2 \leq t - X_3 - \cdots - X_n \mid X_3, \ldots, X_n)),$$

it suffices to prove the result for $n = 2$. Further, it is clear that it suffices to give a proof for integers $t \geq 2$.

For $n = 2$ we have

$$\begin{aligned} F(p_1, p_2, t) &= p_1 p_2 \sum_{k=0}^{[t]-2} \sum_{j=0}^{k} (1-p_1)^j (1-p_2)^{k-j} \\ &= \frac{p_1 p_2}{p_1 - p_2}\left((1-p_2)\frac{1-(1-p_2)^{m+1}}{p_2} - (1-p_1)\frac{1-(1-p_1)^{m+1}}{p_1}\right), \end{aligned} \tag{3}$$

where $m = [t] - 2$. Putting $p_1 = x$, $p_2 = c - x$, we obtain the expression

$$\begin{aligned} f(x) = F(x, c-x, t) = \frac{x(c-x)}{2x-c}\Bigg((1-c+x)\cdot\frac{1-(1-c+x)^{m+1}}{c-x} \\ - (1-x)\cdot\frac{1-(1-x)^{m+1}}{x}\Bigg), \end{aligned}$$

where $m = 0, 1, 2, \ldots$.

We have to show that f is decreasing in $x \in (c/2, c)$. However, the conditions

$$0 \leq x \leq 1, \qquad 0 \leq 1 - c + x \leq 1, \qquad \frac{c}{2} \leq x \leq c, \qquad 0 \leq c \leq 2$$

are met if and only if

$$\frac{c}{2} \leq x \leq \min(c, 1), \qquad 0 \leq c \leq 2. \tag{4}$$

Therefore, we will show that the function f is decreasing in x, for x and c as specified in (4).

We have

$$\begin{aligned} f'(x) &= -\frac{1}{(2x-c)^2}((2x^2(m+2) - cx(m+3) + c(c-1))(x-c+1)^{m+1} \\ &\quad - (2x^2(m+2) - cx(3m+5) + c(c(m+2)-1))(1-x)^{m+1}) \\ &= -\frac{1}{(2x-c)^2} g(x), \end{aligned}$$

where

$$\begin{aligned} g'(x) &= (m+2)(2x-c)((x(m+3) - c(m+1) - 2)(1-x)^m \\ &\quad + (x(m+3) - 2c + 2)(x-c+1)^m) \\ &= (m+2)(2x-c)\varphi(x). \end{aligned}$$

As $g(c/2) = 0$, the proof will be finished if we show that $\varphi(x) \geq 0$. This inequality is equivalent to

$$\alpha(x)(x-c+1)^m \geq \beta(x)(1-x)^m, \tag{5}$$

where

$$\alpha(x) = x(m+3) - 2c + 2, \qquad \beta(x) = 2 + c(m+1) - x(m+3).$$

It is not difficult to show that both α and β are non-negative for x as in (4). Further,

$$\alpha(x) - \beta(x) = (2x-c)(m+3) \geq 0 \tag{6}$$

and, by $x \geq c/2$,

$$(x - c + 1)^m \geq (1 - x)^m. \tag{7}$$

Inequalities (6) and (7) imply (5), which ends the proof. □

THEOREM 2. *Let $X_1, \ldots, X_n$ be independent negative binomial random variables such that*

$$P(X_i = k) = \binom{k-1}{r-1} p_i^r (1 - p_i)^{k-r} \qquad k = r, r+1, \ldots, \quad r \in \mathbf{N}. \tag{8}$$

Then the function

$$F(p_1, p_2, \ldots, p_n; t) = P(X_1 + X_2 + \cdots + X_n \leq t) \tag{9}$$

is Schur-concave with respect to $\mathbf{p}$ *for every real t and for any arbitrary natural number r.*

Proof. Each of the random variables X_i represents time until the r-th success in a sequence of Bernoulli trials. Therefore,

$$X_i = Y_{i,1} + Y_{i,2} + \cdots + Y_{i,r},$$

where $Y_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq r$, are independent geometric random variables. The assertion follows by an application of Theorem 1, noticing that

$$\mathbf{p} \prec \mathbf{p}' \Rightarrow (p_1, p_1, \ldots, p_1, \ldots, p_n, p_n, \ldots, p_n)$$
$$\prec (p'_1, p'_1, \ldots, p'_1, \ldots, p'_n, p'_n, \ldots, p'_n),$$

where each of p_i, p'_i occurs r times. □

Inequality (1) applied to the function F reads

$$F(p_1, p_2, \ldots, p_n; t) \leq F(\bar{p}, \bar{p}, \ldots, \bar{p}; t), \qquad \bar{p} = \frac{p_1 + \cdots + p_n}{n}.$$

Using the fact that the sum of independent negative binomial random variables with the same $\bar{p}$ is also negative binomial, we get the following result.

COROLLARY 1. *Let $X_1, \ldots, X_n$ be independent negative binomial random variables as in Theorem 2, and let $\bar{p} = (p_1 + \cdots + p_n)/n$. Then for each integer $k \geq nr$,*

$$P(X_1 + \cdots + X_n \leq k) \leq \left(\frac{\bar{p}}{1-\bar{p}}\right)^{nr} \sum_{j=nr}^{k} \binom{j-1}{nr-1} (1-\bar{p})^j. \qquad \square$$

Let us consider now independent random variables X_i with negative binomial distributions (8) with $r = r_i$, $i = 1, 2, \ldots, n$. If the r_i's are not equal, the Schur-concavity of $F(p_1, \ldots, p_n)$ cannot be established by reducing the problem to the geometric distribution, because an implication as in (10) does not hold generally. However, some particular inequalities can be derived, like the one in the following theorem.

THEOREM 3. *Let $X_1, X_2, \ldots, X_n$ be independent random variables with*

$$P(X_i = k) = \binom{k-1}{r_i - 1} p_i^{r_i} (1-p_i)^{k-r_i}, \qquad k = r_i, r_i + 1, \ldots, r_i \in \mathbf{N}.$$

Then for every $k \geq r_1 + \cdots + r_n$,

$$F(p_1, \ldots, p_n; k) = P(X_1 + \cdots + X_n \leq k) \leq F(\bar{p}, \ldots, \bar{p}; k) \tag{11}$$

$$= \left(\frac{\bar{p}}{1-\bar{p}}\right)^{R} \sum_{j=R}^{k} \binom{j-1}{R-1} (1-\bar{p})^j, \tag{12}$$

where $R = r_1 + \cdots + r_n$ and $\bar{p} = (r_1 p_1 + \cdots + r_n p_n)/R$.

Proof. As in the proof of Theorem 2, we note that

$$P(X_1 + \cdots + X_n \leq k) = P(Y_{11} + \cdots + Y_{1r_1} + Y_{21} + \cdots + Y_{nr_n} \leq k),$$

where Y_{ij}, $1 \leq i \leq n$, $1 \leq j \leq r_i$ are independent geometric random variables with parameters p_i. By the Schur-concavity result of Theorem 1, we conclude that the expression on the right hand side of the above equality is not greater than

$$P(\bar{Y}_{11} + \cdots + \bar{Y}_{1r_1} + \bar{Y}_{21} + \cdots + \bar{Y}_{nr_n} \leq k),$$

where $\bar{Y}_{ij}$ are independent identically distributed geometric random variables with parameter $\bar{p} = (r_1 p_1 + \cdots + r_n p_n)/(r_1 + \cdots + r_n)$. This proves (11). The equality (12) follows after simplifications as in the proof of Corollary 1. $\square$

Acknowledgement

This research was supported by Science Fund of Serbia, grant number 0401A, through Mathematical Institute, Belgrade.

REFERENCES

[1] BOCK, M. E., DIACONIS, P., HUFFER, F. W. and PERLMAN, M. D., *Inequalities for linear combinations of Gamma random variables.* Canad. J. Statist. *15*, No. 4 (1987), 387–395.

[2] DIACONIS, P. and PERLMAN, M. D., *Bounds for tail probabilities of linear combinations of independent Gamma random variables.* The Symposium on Dependence in Statistics and Probability, Hidden Valley, Pennsylvania, 1987.

[3] GLESER, L., *On the distribution of the number of successes in independent trials.* Ann. Probab. *3* (1975), 182–188.

[4] HOEFFDING, W., *On the distribution of the number of successes in independent trials.* Ann. Math. Statist. *27* (1956), 713–721.

[5] KANTER, M., *Probability inequalities for convex sets and multi-dimensional concentration functions.* J. Multivariate Anal. *6* (1976), 222–236.

[6] MARSHALL, A. and OLKIN, I., *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York, 1979.

[7] PROSCHAN, F., *Peakedness of distributions of convex combinations.* Ann. Math. Statist. *36* (1965), 1703–1706.

[8] TONG, Y. L., *Some inequalities for sums of independent exponential and gamma variables with applications*, unpublished report, Department of Mathematics and Statistics, University of Nebraska, Lincoln, 1980.

M. Merkle
University of Belgrade
Faculty of Electrical Engineering
P.O. Box 816
YU-11001 Belgrade
Yugoslavia

L. Petrović
University of Kragujevac
Faculty of Science
Radoja Domanovića 12
YU-34000 Kragujevac
Yugoslavia