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An inequality for residual of Maclaurin expansion

By

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Introduction. Let $I_n(x)$ be the residual after the n^{th} term in Maclaurin expansion of a function f.

The inequality

$$\frac{I_{n-1}(x)\,I_{n+1}(x)}{I_n^2(x)} \geqq \frac{n+1}{n+2} \tag{1}$$

was proved in [1], for $f(x) = e^x$. This inequality was generalized in [3] and [4].

The left hand side of (1) indicates that the inequality is connected to logarithmic convexity of a certain sequence. In this paper we show that inequality (1) holds for a class of functions f with log-convex derivatives. Further, since

$$I_n(x) = f^{(n+1)}(\xi_n)\frac{x^{n+1}}{(n+1)!}, \quad \xi_n \in (0, x)$$

we have

$$\lim_{x\to 0} \frac{I_{n-1}(x)\,I_{n+1}(x)}{I_n^2(x)} = \frac{f^{(n)}(0)\,f^{(n+2)}(0)}{(f^{(n+1)}(0))^2} \cdot \frac{n+1}{n+2} \tag{2}$$

and this fact opens a question of validity of the inequality

$$\frac{I_{n-1}(x)\,I_{n+1}(x)}{I_n^2(x)} \geqq \frac{f^{(n)}(0)\,f^{(n+2)}(0)}{(f^{(n+1)}(0))^2} \cdot \frac{n+1}{n+2}. \tag{3}$$

Sufficient conditions for (3) to hold were found in [3]. We provide a different sufficient condition in terms of logarithmic convexity.

Assumptions and notations. Let f be an infinitely differentiable function at zero with all derivatives positive, such that its Maclaurin series converges to $f(x)$ for each $x \in (0, R)$, $R > 0$. Let $a_n = f^{(n)}(0)/n!$ be the n-th coefficient of Maclaurin series and let

$$\gamma_n = \frac{a_{n+1}}{a_n} = \frac{f^{(n+1)}(0)}{(n+1)\,f^{(n)}(0)}, \quad I_n(x) = \sum_{k=n+1}^{+\infty} a_k x^k.$$

Let us first prove an auxiliary result.

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Lemma. *Let $\{u_n\}$, $n = 1, 2, \ldots$ be a positive sequence. The inequality*

$$\frac{\varphi(n-1)\,\varphi(n+1)}{\varphi^2(n)} \geqq u_n$$

holds for every $n = 1, 2, \ldots$ if and only if the mapping

$$n \mapsto \alpha(n)\,\varphi(n) \equiv \Psi(n)$$

is log-*convex, where*

$$\alpha(n) = \frac{1}{u_{n-1}\,u_{n-2}^2 \cdots u_1^{n-1}}.$$

Proof. The assertion follows from

$$\frac{\Psi(n-1)\,\Psi(n+1)}{\Psi^2(n)} = \frac{\varphi(n-1)\,\varphi(n+1)}{\varphi^2(n)} \cdot \frac{u_{n-1}^2\,u_{n-2}^4 \cdots u_1^{2n-2}}{u_{n-2}\,u_{n-3}^2 \cdots u_1^{n-2} \cdot u_n\,u_{n-1}^2 \cdots u_1^n}$$
$$= \frac{\varphi(n-1)\,\varphi(n+1)}{\varphi^2(n)} \cdot \frac{1}{u_n}.$$

Theorem 1. *Let the mapping $n \mapsto f^{(n)}(0)$ be* log-*convex (or, equivalently, let $(n+1)\gamma_n$ be an increasing sequence) for $n \geqq n_0$. Then* (1) *holds for every $u \geqq n_0$ and every $x \in (0, R)$.*

Proof. By Lemma, (1) holds if and only if $(n+1)!\,I_n(x)$ is log-convex with respect to n. Now note that

$$(n+1)!\,I_n(x) = x^{n+1} f^{(n+1)}(0) + \sum_{k=2}^{+\infty} b_{n,k}\, x^{n+k}, \tag{4}$$

where $b_{n,k} = \dfrac{f^{(n+k)}(0)}{(n+2)\cdots(n+k)}$, $k = 2, 3, \ldots$.

Since $n \mapsto (n+j)^{-1}$ is log-convex for each j, log-convexity of $f^{(n)}(0)$ implies log-convexity of each summand in (4) and therefore the sum is also log-convex (see [1], for example).

Theorem 2. *Let $n \mapsto \gamma_n$ be* log-*convex for $n \geqq n_0$. Then* (3) *holds for $n \geqq n_0$ and $x \in (0, R)$ and this is the best inequality in the sense that the right hand side of* (3) *can not be replaced by a larger constant.*

Proof. By Lemma, (3) holds if and only if the mapping $n \mapsto (n+1)!\,I_n(x)/f^{(n+1)}(0)$ is log-convex. Since

$$\frac{(n+1)!\,I_n(x)}{f^{(n+1)}(0)} = x^{n+1} + \sum_{k=2}^{+\infty} c_{n,k}\, x^{n+k}, \tag{5}$$

where $c_{n,k} = \dfrac{f^{(n+k)}(0)}{f^{(n+1)}(0)(n+2)\cdots(n+k)}$, we see, in the same way as in the proof of Theorem 1, that the sum (5) will be log-convex if $c_{n,k}$ is log-convex with respect to n for all k, i.e.

$$\frac{c_{n-1,k}\,c_{n+1,k}}{c_{n,k}^2} \geqq 1. \tag{6}$$

It is easy to see that (6) holds if and only if γ_n is log-convex, which ends the proof. From (2) it follows that the right hand side of (3) can not be replaced by a larger constant.

Remarks and examples. In [3] it was proved that (3) holds if γ_n is decreasing and convex. Our condition (γ_n is log-convex) is easier to check and it holds for all examples mentioned in [3]. In particular, for Mittag-Leffler functions

$$E_{1/\lambda}(x) = \sum_{n=0}^{+\infty} \frac{x^n}{\Gamma(1+n/\lambda)}$$

we have

$$\gamma_n = \frac{n\,\Gamma(n/\lambda)}{(n+1)\,\Gamma((n+1)/\lambda)}$$

and this is a log-convex sequence due to the fact that the second deriative of the function $x \mapsto \log x - \log(x+1) + \log \Gamma(x/\lambda) - \log \Gamma((x+1)/\lambda)$ is

$$\sum_{k=1}^{+\infty} \left(\frac{1}{(x+\lambda k)^2} - \frac{1}{(x+1+\lambda k)^2} \right) > 0.$$

Let

$$f(x) = -\sqrt{1-x} = -1 + \sum_{n=1}^{+\infty} \frac{(2n-3)!!}{(2n)!!} x^n.$$

Here we have that $\gamma_n = (2n-1)/(2n+2)$ and this is an increasing sequence. Neither condition from [3] nor the condition of Theorem 2 are satisfied. However, the condition of Theorem 1 holds; therefore, (1) is valid for this function.

For a hypergeometric function

$$F(a,b,c;x) = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{n!\,(c)_n} x^n$$

we have that

$$\gamma_n = \frac{(a+n)(b+n)}{(n+1)(c+n)}.$$

If $a \geqq 1$, $b \geqq c$, this expression is log-convex and (3) holds.

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