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AN INEQUALITY INVOLVING *ERF* AND HYPERBOLIC FUNCTIONS

by

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In this note we derive an inequality involving the function

$$\text{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,\mathrm{d}u \tag{1}$$

Let us start with an auxilliary result, which can be of some interest by itself.
Lemma. Let

$$F(x)=\int_x^{\infty} e^{-t^2}\,\mathrm{d}t. \tag{2}$$

Then for every $u\in\boldsymbol{R}, v\geq 0$

$$F(u-v)\geq e^{4uv}F(u+v) \tag{3}$$

Proof. Introducing $z=t+2v$, we have:

$$F(u-v)=\int_{u-v}^{+\infty} e^{-t^2}\,\mathrm{d}t=\int_{u+v}^{+\infty} e^{-(z-2v)^2}\,\mathrm{d}z \tag{4}$$

$$=\int_{u+v}^{+\infty} e^{-z^2}e^{4zv-4v^2}\,\mathrm{d}z>e^{4uv}\int_{u+v}^{+\infty} e^{-z^2}\,\mathrm{d}z$$
$$=e^{4uv}F(u+v),$$

where we used the inequality

$$4zv-4v^2\geq 4(u+v)v-4v^2=4uv,$$

that holds for $z\geq u+v, v\geq 0$.

Theorem. For every real t and α of the same sign,

$$\frac{1}{2}\left(e^t\text{erf}\left(\alpha t+\frac{1}{2\alpha}\right)+e^{-t}\text{erf}\left(\alpha t-\frac{1}{2\alpha}\right)\right)\geq \sinh t \tag{5}$$

Proof. The inequality (5) is equivalent to

$$(6)\qquad e^{-t}\left(1+\operatorname{erf}\left(\alpha t-\frac{1}{2\alpha}\right)\right)\geq e^{t}\left(1-\operatorname{erf}\left(\alpha t+\frac{1}{2\alpha}\right)\right)$$

Further,

$$1+\operatorname{erf}\left(\alpha t-\frac{1}{2\alpha}\right)=\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\alpha t-\frac{1}{2\alpha}} e^{-u^2}\,\mathrm{d}u=\frac{2}{\sqrt{\pi}}\int_{\frac{1}{2\alpha}-\alpha t}^{+\infty} e^{-u^2}\,\mathrm{d}u=\frac{2}{\sqrt{\pi}}F\left(\frac{1}{2\alpha}-\alpha t\right),$$

where F was introduced in the Lemma. Also

$$1-\operatorname{erf}\left(\alpha t+\frac{1}{2\alpha}\right)=\frac{2}{\sqrt{\pi}}\int_{\alpha t+\frac{1}{2\alpha}}^{+\infty} e^{-u^2}\,\mathrm{d}u=\frac{2}{\sqrt{\pi}}F\left(\frac{1}{2\alpha}+\alpha t\right).$$

Therefore, (6) reduces to

$$(7)\qquad F\left(\frac{1}{2\alpha}-\alpha t\right)\geq e^{2t}F\left(\frac{1}{2\alpha}+\alpha t\right)$$

Introducing $u=\frac{1}{2\alpha}$, $v=\alpha t$, we have $2t=4uv$, and (7) becomes

$$F(u-v)\geq e^{4uv}F(u+v),$$

which was proved in the Lemma.

To see how sharp (5) is, we worked out some examples in the table below on a computer using a FORTRAN program. For small values of α, the inequality is very sharp.

α	t	Error
0.10	1.00	0.000000
0.10	3.00	0.000000
0.10	5.00	0.000000
0.50	1.00	0.042131
0.50	3.00	0.033763
0.50	5.00	0.006577
1.00	1.00	0.233613
1.00	3.00	0.049769
1.00	5.00	0.006744
3.00	1.00	0.367858
3.00	3.00	0.049788
3.00	5.00	0.006744

Table: Errors in (5) for some values of α and t.

Inequality (5) has an application in a problem appearing in Automatic Control Theory. As pointed out in [1], the integral

$$J=\int_{-\infty}^{+\infty}\frac{x\sin\lambda x}{x^2+\beta^2}e^{\frac{\sigma^2-x^2}{2}}\,\mathrm{d}x$$

appears in an expression for the error probability of a binary information system. We shall show in another paper that for $\lambda>0$, the inequality $J\geq 0$ is equivalent to the inequality of our Theorem, with $t=\beta\lambda$ and $\alpha=1/\sqrt{2}\beta\sigma$.

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Reference

[1] **V. K. Prabhu** - *Modified Chernoff bounds for PAM systems with noise and interference*, IEEE trans. on inform. theory, vol. IT-28, No 1, pp 95-99, 1982.