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SIGNAL AVERAGING IN FOURIER-TRANSFORM SPECTROSCOPY (TWO-SIDED INTERFEROGRAM)

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Abstract—In this paper we discuss two types of data averaging in FTS: the averaging of interferograms and the averaging of spectral moduli. We investigated relations between mathematical expectations as well as between standard deviations in both cases for different number of averaged terms. These results are based on our previous paper *Infrared Phys.* **27,** 297, 1987.

For two-sided interferograms the following relation holds:

$$B = (C^2 + S^2)^{1/2}, \tag{1}$$

where C and S are cosine and sine transforms of a two-sided interferogram. Under the assumption of the Gaussian interferogram noise, the expected value m_B and the variance σ_B^2 of B are given by:[1]

$$m_B = E(B) = \sigma f(a), \tag{2}$$

$$\sigma_B = (\mathrm{var}(B))^{1/2} = \sigma g(a), \tag{3}$$

where:

$$f(a) = \left(\frac{\pi}{2}\right)^{1/2} \exp\left(-\frac{a^2}{2}\right) M\left(\frac{3}{2}, 1, \frac{a^2}{2}\right), \tag{4}$$

$$g(a) = (a^2 + 2 - f^2(a))^{1/2}, \tag{5}$$

$$\sigma^2 = \mathrm{var}(C) = \mathrm{var}(S), \tag{6}$$

$$a = b/\sigma, \tag{7}$$

$$b = (E^2(C) + E^2(S))^{1/2}, \tag{8}$$

$M(a, b, z)$ is the Confluent Hypergeometric function.[2] The averaging of interferograms is equivalent to averaging of both cosine and sine transforms taken separately,[3,4] due to linear relations between transforms and an interferogram. Therefore the spectrum obtained from the averaged interferogram is:

$$\underline{B} = (\bar{C}^2 + \bar{S}^2)^{1/2}, \tag{9}$$

where $\bar{C}$ and $\bar{S}$ are averaged transforms with standard deviations:

$$\sigma_{\bar{C}} = \sigma_{\bar{S}} = \bar{\sigma} = \frac{\sigma}{n^{1/2}}, \tag{10}$$

where n is the number of averaged terms.

From equations (2), (3) and (10) it follows:

$$m_{\underline{B}} = \bar{\sigma} \cdot f(\bar{a}) = \frac{\sigma}{n^{1/2}} f(an^{1/2}), \tag{11}$$

$$\sigma_{\underline{B}} = \bar{\sigma} \cdot g(\bar{a}) = \frac{\sigma}{n^{1/2}} g(an^{1/2}). \tag{12}$$

The averaged modulus is defined as:

$$\bar{B} = \frac{1}{n} \sum_{k=1}^{n} B_k. \tag{13}$$

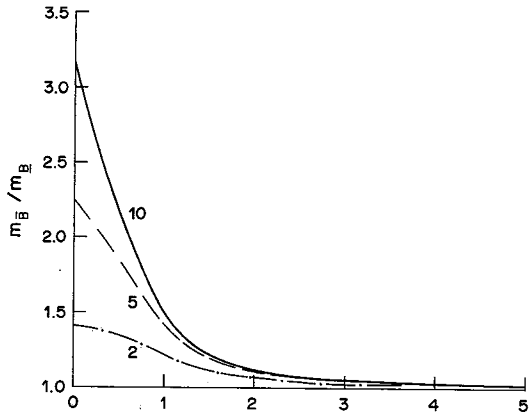

Fig. 1. The ratio of mathematical expectations $m_{\bar{B}}/m_{\underline{B}}$, as a function of a, for number of averaged terms $n = 2, 5, 10$.

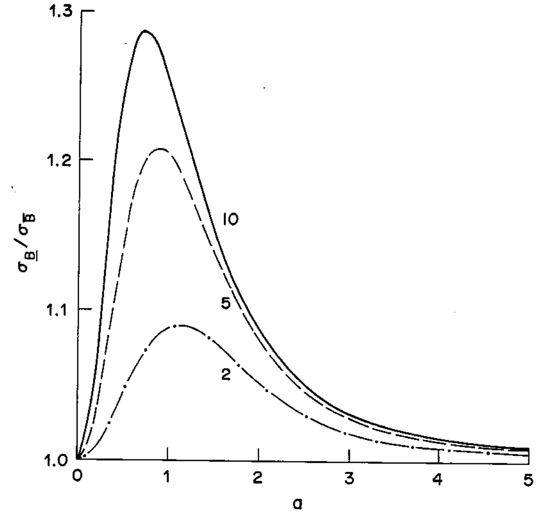

Fig. 2. The ratio of standard deviations $\sigma_{\underline{B}}/\sigma_{\bar{B}}$ as a function of a, for number of averaged terms $n = 2, 5, 10$.

So, we obtain:

$$m_{\bar{B}} = \frac{1}{n}\sum_{k=1}^{n} m_{B_k} = m_B = \sigma f(a), \tag{14}$$

$$\sigma_{\bar{B}} = \frac{\sigma_B}{n^{1/2}} = \frac{\sigma}{n^{1/2}} g(a). \tag{15}$$

Now, we shall compare expectations and standard deviations for two types of data averaging.

By studying the features of the functions f and g we come to the following conclusion:

$$m_{\underline{B}} < m_{\bar{B}}, \tag{16}$$

$$\sigma_{\underline{B}} > \sigma_{\bar{B}}. \tag{17}$$

(The proof of these relations is provided in the Appendix.) These relations show that signal level of the averaged modulus is always higher than the level of the spectrum obtained by the interferogram averaging, whereas the noise level of spectrum in the former case is always smaller than in the latter case.

The following figures present ratios $m_{\bar{B}}/m_{\underline{B}}$ and $\sigma_{\underline{B}}/\sigma_{\bar{B}}$ when a varies for different values of n. This analysis is particularly useful in cases when the signal to noise ratio a is low (e.g. photoacoustic FTIR). From Fig. 2 one can see that the ratio $\sigma_{\underline{B}}/\sigma_{\bar{B}}$ is about 1.3 for $n = 10$. At the same time, the ratio $m_{\bar{B}}/m_{\underline{B}}$ is about 1.8, which is the price one pays for lowering the noise level.

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APPENDIX

We shall first prove equation (16). Let f be defined as in equation (4). By substituting equations (11) and (14) into equation (16), and introducing:

$$x = an^{1/2} \quad \text{and} \quad \lambda = \frac{1}{n^{1/2}},$$

we get equation (16) in a more general form:

$$\lambda f(x) < f(\lambda x), \quad \text{for all} \quad \lambda \in (0, 1), \tag{A1}$$

i.e.

$$\lambda \exp\left(-\frac{x^2}{2}\right) M\left(\frac{x^2}{2}\right) < \exp\left(-\frac{\lambda^2 x^2}{2}\right) M\left(\frac{\lambda^2 x^2}{2}\right) \tag{A2}$$

where

$$M(z) = M(\tfrac{3}{2}, 1, z). \tag{A3}$$

If we define:

$$t = \frac{x^2}{2} \quad \text{and} \quad \eta = \lambda^2,$$

then equation (1) reduces to:

$$\frac{\exp(-t)M(t)}{t^{1/2}} < \frac{\exp(-\eta t)M(\eta t)}{(\eta t)^{1/2}} \quad \text{for all} \quad \eta \in (0, 1) \tag{A4}$$

To show equation (A4) it suffices to prove that the function ϕ defined by:

$$\phi(t) = \frac{\exp(-t)M(t)}{t^{1/2}}, \tag{A5}$$

is decreasing.

The first derivative of ϕ is given by:

$$\phi'(t) = \frac{\exp(-t)}{2 \cdot t^{3/2}} (2t(M'(t) - M(t)) - M(t)). \tag{A6}$$

The series representation of $M(t)$ is:[(2)]

$$M(t) = 1 + \sum_{k=1}^{+\infty} \frac{(2k+1)!!}{2^k (k!)^2} t^k. \tag{A7}$$

From equations (A6) and (A7) one can easily show that $\phi'(t) < 0$, so $\phi(t)$ is decreasing, which had to be proved.

According to equation (12) and (15), the relation (17) is a consequence of the fact that the function g is increasing. We have:

$$g^2(x) = x^2 + 2 - f^2(x) = 2 + t\left(2 - \frac{\pi}{2}\phi^2(t)\right). \tag{A8}$$

Since ϕ is proved to be decreasing, it is straightforward to conclude that g is increasing,which ends the proof.