Inequalities for Residuals of Power Expansions for the Exponential Function and Completely Monotone Functions*

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Let $I_n(x)$ be the residual after the $n$th term of power series for a function $f$. In the case $f(x) = e^x$, $x > 0$, inequalities for $I_n$ have been studied by many authors, but there are almost no results for $x < 0$. In this paper we present some bounds for $I_n$ with $f(x) = e^x$ and we give some results for completely monotone functions.

1. INTRODUCTION

Let $I_n(x)$ be the residual after the $n$th term of power series for the function $x \rightarrow e^x$. Investigation of bounds for $I_n(x)$ (and generalizations of such inequalities to other functions) has attracted the attention of a considerable number of mathematicians (see the references). We were able to trace the interest in the topic back to 1911 (Ramanujan [24]); it was revived by some recent contributions [2, 3, 5, 8, 16–19] connected with Kesava Menon’s inequality [15]. However, with a very few exceptions the case $x < 0$ has not been treated.

Let

$$I_n(x) = e^{-x} - \sum_{k=0}^{n} \frac{(-1)^k x^k}{k!} = \sum_{k=n+1}^{\infty} \frac{(-1)^k x^k}{k!}, \quad x > 0. \quad (1)$$

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In Sections 2 and 3 we prove that
\[ \frac{n}{n+1} \leq \frac{I_{n-1}(x)I_{n+1}(x)}{I_n^2(x)} \leq \frac{n+1}{n+2} \]
and
\[ |I_n(x)| = \frac{x^{n+1}}{(n+1)!\left(1 + x/(n + \theta)\right)}, \quad 1 < \theta < 2. \]
In Section 4 we discuss the residual (1) for completely monotone functions.

2. KEŠAVA MENON’S TYPE INEQUALITIES FOR $I_n$

In 1943, Kesava Menon [15] obtained the inequality
\[ \frac{I_{n-1}(x)I_{n+1}(x)}{I_n^2(x)} > \frac{1}{2}, \quad x > 0, n = 1, 2, \ldots, \]
where $I_n$ is the residual after the $n$th term in the Maclaurin expansion for $x \mapsto e^x$, $x > 0$. The following bounds are given in [2; 17, Lemma 3], respectively,
\[ \frac{n+1}{n+2} < \frac{I_{n-1}(x)I_{n+1}(x)}{I_n^2(x)} < 1. \]
Inequalities of this type were considered also in [3, 5, 8, 18, 19]. Here we give bounds for the ratio in (2), with $I_n$ for $x \mapsto e^{-x}$, $x > 0$, as defined by (1).

We start with the following simple, but useful lemma.

**Lemma 1.** Let $I_n$ be as defined in (1). Then

(i) $I_n'(x) = -I_{n-1}(x)$,

(ii) $\text{sgn } I_n(x) = (-1)^{n+1}$,

(iii) $x \mapsto |I_n(x)|$ is an $n$-monotone function on $[0, +\infty)$, as defined in [10], i.e.,
\[ g^{(k)}(x) \geq 0 \quad \text{for } x > 0, k = 0, 1, \ldots, n+1. \]

**Proof.** Part (i) is easy to prove by differentiation. From the integral form of the residual in Maclaurin’s expansion we have that
\[ (-1)^{n+1}I_n(x) = \frac{1}{n!} \int_0^x (x - t)^n e^{-t} \, dt, \quad n = 0, 1, \ldots \quad (3) \]
and therefore \((-1)^{n+1} I_n(x) > 0\) for \(x > 0\), which proves (ii). By [10], if \(\nu\) is a regular Borel measure, then the function

\[ g(x) = \frac{1}{n!} \int_0^x (x-t)^n \, d\nu(t) \]

is \(n\) monotone, and (iii) is proved.

The next result is a counterpart to (2) in the case of a negative argument of the exponential function.

**Theorem 1.** For \(n = 1, 2, \ldots\) and \(x \in (0, +\infty)\) we have

\[ \frac{n}{n+1} \leq \frac{I_{n-1}(x)}{I_n(x)} \leq \frac{n+1}{n+2}. \]  

(4)

Bounds in (4) are the best possible constant bounds for \(x \in (0, +\infty)\).

**Proof.** By [10, Theorem 4], if \(g\) is an \(n + 1\)-monotone function on \([0, T]\), then

\[ g''(x)g(x) \geq \frac{n}{n+1} g'^2(x), \quad x \in [0, T]. \]

An application of this inequality to \(g(x) = (-1)^n I_{n+1}(x)\), together with Lemma 1, yields the left inequality in (4).

To prove the right inequality in (4), we note that

\[ |I_n(x)| = \frac{x^{n+1}}{(n+1)!} \left( 1 - \frac{x}{n+2} + \frac{x^2}{(n+2)(n+3)} - \cdots \right) \]

\[ = \frac{x^{n+1}}{(n+1)!} M(1, n+2, -x), \]  

(5)

where \(M\) is the confluent hypergeometric function. By Kummer's transformation [1, 13.1.27], we have that

\[ M(1, n+2, -x) = e^{-x} M(n+1, n+2, x) = e^{-x} \left( 1 + \sum_{k=1}^{\infty} \frac{n+1}{(n+k+1)k} x^k \right) \]
and therefore

$$|I_n(x)| = \frac{x^{n+1}}{n!} e^{-x} \sum_{k=0}^{+\infty} \frac{x^k}{(n+k+1)k!}, \quad n = 0, 1, 2, \ldots \quad (6)$$

Now we will show that the mapping \( n \mapsto (n + 1)! |I_n(x)| \) is log-concave for each fixed \( x \in (0, +\infty) \). Indeed, by (6) we have that

$$(n + 1)! |I_n(x)| = x^{n+1} e^{-x} \sum_{k=0}^{+\infty} \frac{n + 1}{(n+k+1)k!} x^k, \quad (7)$$

where the mapping \( n \mapsto x^{n+1} e^{-x} \) is log-concave for each \( x > 0 \). The mapping \( n \mapsto (n + 1)/(n + k + 1) \) is concave (by formal differentiation with respect to \( n \)) and therefore the sum in (7) is also concave and consequently log-concave. Since the product of two log-concave mappings is also log-concave, we conclude that \( n \mapsto (n + 1)! |I_n(x)| \) is log-concave. This is equivalent to the right inequality in (4).

Let us now show that the constant bounds in (4) are the best possible. From (5) it follows that

$$\frac{I_{n-1}(x)I_{n+1}(x)}{I_n^2(x)} = \frac{n + 1}{n + 2} \frac{M(1, n + 1, -x)M(1, n + 3, -x)}{M^2(1, n + 2, -x)}.$$  

Since \( \lim_{x \to 0} M(1, n, -x) = 1 \) for all \( n \), we have that

$$\lim_{x \to 0} \frac{I_{n-1}(x)I_{n+1}(x)}{I_n^2(x)} = \frac{n + 1}{n + 2}$$

and the upper bound in (4) cannot be replaced by a smaller constant.

Further, by [1, 13.1.5] we have that

$$M(1, n, -x) = \frac{n - 1}{x} (1 + O(1/x)) \quad (x \to +\infty) \quad (8)$$

and so

$$\lim_{x \to +\infty} \frac{I_{n-1}(x)I_{n+1}(x)}{I_n^2(x)} = \frac{n + 1}{n + 2} \cdot \frac{n(n + 2)}{(n + 1)^2} = \frac{n}{n + 1}.$$  

Therefore, the lower bound in (4) is the best possible.
3. BOUNDS FOR $|I_n|$  

**Theorem 2.** For each $x \in (0, +\infty)$ and each nonnegative integer $n$ there exists a $\theta \in (1,2)$ such that

$$|I_n(x)| = \frac{x^{n+1}}{(n+1)!\left(1 + x/(n + \theta)\right)},$$  

(9)

Moreover, for all $n$ we have that $\lim_{x \to 0} \theta = 2$, $\lim_{x \to +\infty} \theta = 1$.

**Proof.** For fixed $x$ and $n$, let $\theta$ be as defined by (9). Then $\theta > 1$ is equivalent to

$$|I_n(x)| > \frac{x^{n+1}}{n!(x + n + 1)},$$

or, using (5),

$$\frac{x + n + 1}{n + 1} M(1, n + 2, -x) > 1.$$  

(10)

Further, we have that

$$\frac{x}{n + 1} M(1, n + 2, -x)$$

$$= \frac{x}{n + 1} \left(1 - \frac{x}{n + 2} + \frac{x^2}{(n + 2)(n + 3)} - \cdots\right)$$

$$= \frac{x}{n + 1} - \frac{x^2}{(n + 1)(n + 2)} + \frac{x^3}{(n + 1)(n + 2)(n + 3)} - \cdots$$

$$= 1 - M(1, n + 1, -1)$$

and (10) is equivalent to $1 - M(1, n + 1, -x) + M(1, n + 2, -x) > -1$, i.e.,

$$M(1, n + 2, -x) > M(1, n + 1, -x), \quad n = 0, 1, \ldots.$$  

(11)

By Kummer's transformation, (11) becomes

$$M(n + 1, n + 2, x) > M(n, n + 1, x),$$

i.e.,

$$\sum_{k=0}^{+\infty} \frac{n + 1}{(n + k + 1)k!} x^k > \sum_{k=0}^{+\infty} \frac{n}{(n + k)k!} x^k,$$
which is true by \((n + 1)/(n + k + 1) > n/(n + k), k > 0\). Therefore, we proved that \(\theta > 1\). The second part can be proved basically in the same way. The inequality \(\theta < 2\) is equivalent to

\[
\left(1 + \frac{x}{n + 2}\right)M(1, n + 2, -x) < 1
\]

and further

\[
1 - \frac{x}{n + 2}M(1, n + 3, -x) + \frac{x}{n + 2}M(1, n + 2, -x) < 1,
\]

i.e.,

\[
M(1, n + 3, -x) > M(1, n + 2, -x), \quad n = 0, 1, \ldots
\]

which is true by (11).

By (9) and (5) we have that

\[
M(1, n + 2, -x) = \frac{1}{1 + x/(n + \theta)}.
\]

Using Maclaurin’s expansion we obtain

\[
1 - \frac{x}{n + 2} + \frac{x^2}{(n + 2)(n + 3)} + \cdots = 1 - \frac{x}{n + \theta} + \frac{x^2}{(n + \theta)^2} - \cdots,
\]

from where it follows that \(\lim_{x \to 0} \theta = 2\). Further, by (8) we have that

\[
\frac{n + 1}{x}(1 + O(1/x)) = \frac{1}{1 + x/(n + \theta)}, \quad (x \to +\infty),
\]

i.e.,

\[
(n + 1)(1 + O(1/x)) = \frac{x}{n + \theta + x}(n + \theta), \quad (x \to +\infty).
\]

Letting \(x \to +\infty\) we get \(\lim_{x \to +\infty} \theta = 1\).
4. INEQUALITIES FOR COMPLETELY MONOTONE FUNCTIONS

Some results presented in previous sections can be generalized for completely monotone functions. Recall that the function $f$ defined on $(0, +\infty)$ is called completely monotone if $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in (0, +\infty)$. By [27, p. 160] or [10], an infinitely differentiable and continuous at 0 function $f$ is completely monotone on $(0, +\infty)$ if and only if

$$f(x) = \int_0^{+\infty} e^{-xt} \, d\alpha(t), \quad (12)$$

where $t \mapsto \alpha(t)$ is bounded and nondecreasing and the integral converges for $0 \leq x < +\infty$.

Suppose that the Maclaurin’s series for $f$ converges to $f(x)$ for $x \in (0, R)$, $R > 0$ and let

$$I_n(x) = \sum_{k=n+1}^{+\infty} \frac{f^{(k)}(0)}{k!} x^k. \quad (13)$$

Then by the integral representation we have that

$$(-1)^{n+1} I_n(x) = \frac{1}{n!} \int_0^x (x-t)^n f^{(n+1)}(t) \, dt > 0, \quad n = 0, 1, \ldots$$

which implies that (ii) and (iii) of Lemma 1 hold for $x \in (0, R)$. Moreover, the following partial generalization of Theorem 1 holds:

**Theorem 3.** If $f$ is a completely monotone function on $(0, +\infty)$ such that its Maclaurin series converges to $f(x)$ for $x \in (0, R)$, then

$$\frac{I_{n-1}(x) I_{n+1}(x)}{I_n^2(x)} \geq \frac{n}{n+1} \quad (14)$$

for each $x \in (0, R)$ and $n = 0, 1, 2, \ldots$.

**Proof.** It is easy to see that (13) is equivalent to the statement that the mapping $n \mapsto (-1)^{n+1} n! I_n(x)$ is log-convex for each $x \in (0, R)$. Let $I_n(t, x)$ denote the residual after the $n$th term in Maclaurin’s expansion of the function $x \mapsto e^{-xt}$ for a $t \in (0, +\infty)$. Then by (12) and the uniqueness theorem for Maclaurin’s expansion, we have that

$$I_n(x) = \int_0^{+\infty} I_n(t, x) \, d\alpha(t). \quad (14)$$
Now from Theorem 1 it follows that the mapping \( n \mapsto (-1)^{n+1} n! I_n(t, x) \) is log-convex for each \( x, t \in (0, +\infty) \) and so is the mapping

\[
n \mapsto (-1)^{n+1} n! \sum_{i=1}^{k} C_i I_n(t_i, x),
\]

where \( C_i, t_i \) are arbitrary positive numbers. By passing to a limit we conclude that the mapping

\[
n \mapsto (-1)^{n+1} n! \int_0^{+\infty} I_n(t, x) \, d\alpha(t)
\]

is also log convex, and (13) follows.

The right inequality in (4) of Theorem 2 cannot be extended to all completely monotone functions. As an example, take \( f(x) = 1/(1 + x), -1 < x < 1. \) Here we have \( |I_n| = |x|^{n+1}/(1 + x), I_{n-1}I_{n+1}/I_n^2 = 1 \) and so the right inequality in (4) does not hold.

**REFERENCES**