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Inequalities for Residuals of Power Expansions for the Exponential Function and Completely Monotone Functions*

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Let $I_n(x)$ be the residual after the nth term of power series for a function f. In the case $f(x) = e^x$, $x > 0$, inequalities for I_n have been studied by many authors, but there are almost no results for $x < 0$. In this paper we present some bounds for I_n with $f(x) = e^{-x}$ and we give some results for completely monotone functions. © 1997 Academic Press

1. INTRODUCTION

Let $I_n(x)$ be the residual after the nth term of power series for the function $x \mapsto e^x$. Investigation of bounds for $I_n(x)$ (and generalizations of such inequalities to other functions) has attracted the attention of a considerable number of mathematicians (see the references). We were able to trace the interest in the topic back to 1911 (Ramanujan [24]); it was revived by some recent contributions [2, 3, 5, 8, 16–19] connected with Kesava Menon's inequality [15]. However, with a very few exceptions the case $x < 0$ has not been treated.

Let

$$I_n(x) = e^{-x} - \sum_{k=0}^{n} (-1)^k \frac{x^k}{k!} = \sum_{k=n+1}^{+\infty} (-1)^k \frac{x^k}{k!}, \qquad x > 0. \tag{1}$$

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In Sections 2 and 3 we prove that

$$\frac{n}{n+1} \leq \frac{I_{n-1}(x)I_{n+1}(x)}{I_n^2(x)} \leq \frac{n+1}{n+2}$$

and

$$|I_n(x)| = \frac{x^{n+1}}{(n+1)!(1+x/(n+\theta))}, \qquad 1 < \theta < 2.$$

In Section 4 we discuss the residual (1) for completely monotone functions.

2. KESAVA MENON'S TYPE INEQUALITIES FOR I_n

In 1943, Kesava Menon [15] obtained the inequality

$$\frac{I_{n-1}(x)I_{n+1}(x)}{I_n^2(x)} > \frac{1}{2}, \qquad x > 0, n = 1, 2, \ldots,$$

where I_n is the residual after the nth term in the Maclaurin expansion for $x \mapsto e^x$, $x > 0$. The following bounds are given in [2; 17, Lemma 3], respectively,

$$\frac{n+1}{n+2} < \frac{I_{n-1}(x)I_{n+1}(x)}{I_n^2(x)} < 1.$$

Inequalities of this type were considered also in [3, 5, 8, 18, 19]. Here we give bounds for the ratio in (2), with I_n for $x \mapsto e^{-x}$, $x > 0$, as defined by (1).

We start with the following simple, but useful lemma.

LEMMA 1. *Let I_n be as defined in* (1). *Then*

(i) $I_n'(x) = -I_{n-1}(x)$,

(ii) $\operatorname{sgn} I_n(x) = (-1)^{n+1}$,

(iii) $x \mapsto |I_n(x)|$ *is an n-monotone function on* $[0, +\infty)$, *as defined in* [10], *i.e.*,

$$g^{(k)}(x) \geq 0 \qquad \text{for } x > 0, k = 0, 1, \ldots, n+1.$$

Proof. Part (i) is easy to prove by differentiation. From the integral form of the residual in Maclaurin's expansion we have that

$$(-1)^{n+1} I_n(x) = \frac{1}{n!}\int_0^x (x-t)^n e^{-t}\, dt, \qquad n = 0, 1, \ldots \tag{3}$$

and therefore $(-1)^{n+1}I_n(x) > 0$ for $x > 0$, which proves (ii). By [10], if v is a regular Borel measure, then the function

$$g(x) = \frac{1}{n!}\int_0^x (x-t)^n \, dv(t)$$

is n monotone, and (iii) is proved. ∎

The next result is a counterpart to (2) in the case of a negative argument of the exponential function.

THEOREM 1. *For* $n = 1, 2, \dots$ *and* $x \in (0, +\infty)$ *we have*

$$\frac{n}{n+1} \le \frac{I_{n-1}(x) I_{n+1}(x)}{I_n^2(x)} \le \frac{n+1}{n+2}. \tag{4}$$

Bounds in (4) *are the best possible constant bounds for* $x \in (0, +\infty)$.

Proof. By [10, Theorem 4], if g is an $n+1$-monotone function on $[0, T]$, then

$$g''(x)g(x) \ge \frac{n}{n+1} g'^2(x), \qquad x \in [0, T].$$

An application of this inequality to $g(x) = (-1)^n I_{n+1}(x)$, together with Lemma 1, yields the left inequality in (4).

To prove the right inequality in (4), we note that

$$\begin{aligned} |I_n(x)| &= \frac{x^{n+1}}{(n+1)!}\left(1 - \frac{x}{n+2} + \frac{x^2}{(n+2)(n+3)} - \cdots\right) \\ &= \frac{x^{n+1}}{(n+1)!} M(1, n+2, -x), \end{aligned} \tag{5}$$

where M is the confluent hypergeometric function. By Kummer's transformation [1, 13.1.27], we have that

$$\begin{aligned} &M(1, n+2, -x) \\ &\quad = e^{-x} M(n+1, n+2, x) = e^{-x}\left(1 + \sum_{k=1}^{+\infty} \frac{n+1}{(n+k+1)k!} x^k\right) \end{aligned}$$

and therefore

$$|I_n(x)| = \frac{x^{n+1}}{n!}e^{-x}\sum_{k=0}^{+\infty}\frac{x^k}{(n+k+1)k!}, \qquad n = 0, 1, 2, \dots . \tag{6}$$

Now we will show that the mapping $n \mapsto (n+1)!|I_n(x)|$ is log-concave for each fixed $x \in (0, +\infty)$. Indeed, by (6) we have that

$$(n+1)!|I_n(x)| = x^{n+1}e^{-x}\sum_{k=0}^{+\infty}\frac{n+1}{(n+k+1)k!}x^k, \tag{7}$$

where the mapping $n \mapsto x^{n+1}e^{-x}$ is log-concave for each $x > 0$. The mapping $n \mapsto (n+1)/(n+k+1)$ is concave (by formal differentiation with respect to n) and therefore the sum in (7) is also concave and consequently log-concave. Since the product of two log-concave mappings is also log-concave, we conclude that $n \mapsto (n+1)!|I_n(x)|$ is log-concave. This is equivalent to the right inequality in (4).

Let us now show that the constant bounds in (4) are the best possible. From (5) it follows that

$$\frac{I_{n-1}(x)I_{n+1}(x)}{I_n^2(x)} = \frac{n+1}{n+2}\cdot\frac{M(1, n+1, -x)M(1, n+3, -x)}{M^2(1, n+2, -x)}.$$

Since $\lim_{x\to 0}M(1, n, -x) = 1$ for all n, we have that

$$\lim_{x\to 0}\frac{I_{n-1}(x)I_{n+1}(x)}{I_n^2(x)} = \frac{n+1}{n+2}$$

and the upper bound in (4) cannot be replaced by a smaller constant. Further, by [1, 13.1.5] we have that

$$M(1, n, -x) = \frac{n-1}{x}(1 + O(1/x)) \qquad (x \to +\infty) \tag{8}$$

and so

$$\lim_{x\to +\infty}\frac{I_{n-1}(x)I_{n+1}(x)}{I_n^2(x)} = \frac{n+1}{n+2}\cdot\frac{n(n+2)}{(n+1)^2} = \frac{n}{n+1}.$$

Therefore, the lower bound in (4) is the best possible. ∎

3. BOUNDS FOR $|I_n|$

THEOREM 2. *For each* $x \in (0, +\infty)$ *and each nonnegative integer* n *there exists a* $\theta \in (1, 2)$ *such that*

$$|I_n(x)| = \frac{x^{n+1}}{(n+1)!(1 + x/(n+\theta))}. \tag{9}$$

Moreover, for all n *we have that* $\lim_{x \to 0} \theta = 2$, $\lim_{x \to +\infty} \theta = 1$.

Proof. For fixed x and n, let θ be as defined by (9). Then $\theta > 1$ is equivalent to

$$|I_n(x)| > \frac{x^{n+1}}{n!(x+n+1)},$$

or, using (5),

$$\frac{x+n+1}{n+1} M(1, n+2, -x) > 1. \tag{10}$$

Further, we have that

$$\begin{aligned}
&\frac{x}{n+1} M(1, n+2, -x) \\
&\quad = \frac{x}{n+1}\left(1 - \frac{x}{n+2} + \frac{x^2}{(n+2)(n+3)} - \cdots\right) \\
&\quad = \frac{x}{n+1} - \frac{x^2}{(n+1)(n+2)} + \frac{x^3}{(n+1)(n+2)(n+3)} - \cdots \\
&\quad = 1 - M(1, n+1, -1)
\end{aligned}$$

and (10) is equivalent to $1 - M(1, n+1, -x) + M(1, n+2, -x) > -1$, i.e.,

$$M(1, n+2, -x) > M(1, n+1, -x), \qquad n = 0, 1, \ldots. \tag{11}$$

By Kummer's transformation, (11) becomes

$$M(n+1, n+2, x) > M(n, n+1, x),$$

$$\text{i.e.,} \ \sum_{k=0}^{+\infty} \frac{n+1}{(n+k+1)k!} x^k > \sum_{k=0}^{+\infty} \frac{n}{(n+k)k!} x^k,$$

which is true by $(n+1)/(n+k+1) > n/(n+k)$, $k > 0$. Therefore, we proved that $\theta > 1$. The second part can be proved basically in the same way. The inequality $\theta < 2$ is equivalent to

$$\left(1 + \frac{x}{n+2}\right) M(1, n+2, -x) < 1$$

and further

$$1 - \frac{x}{n+2} M(1, n+3, -x) + \frac{x}{n+2} M(1, n+2, -x) < 1,$$

i.e.,

$$M(1, n+3, -x) > M(1, n+2, -x), \qquad n = 0, 1, \ldots$$

which is true by (11).

By (9) and (5) we have that

$$M(1, n+2, -x) = \frac{1}{1 + x/(n+\theta)}.$$

Using Maclaurin's expansion we obtain

$$1 - \frac{x}{n+2} + \frac{x^2}{(n+2)(n+3)} + \cdots$$
$$= 1 - \frac{x}{n+\theta} + \frac{x^2}{(n+\theta)^2} - \cdots,$$

from where it follows that $\lim_{x \to 0} \theta = 2$. Further, by (8) we have that

$$\frac{n+1}{x}(1 + O(1/x)) = \frac{1}{1 + x/(n+\theta)} \qquad (x \to +\infty),$$

i.e.,

$$(n+1)(1 + O(1/x)) = \frac{x}{n + \theta + x}(n+\theta) \qquad (x \to +\infty).$$

Letting $x \to +\infty$ we get $\lim_{x \to +\infty} \theta = 1$. ∎

4. INEQUALITIES FOR COMPLETELY MONOTONE FUNCTIONS

Some results presented in previous sections can be generalized for completely monotone functions. Recall that the function f defined on $(0, +\infty)$ is called completely monotone if $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in (0, +\infty)$. By [27, p. 160] or [10], an infinitely differentiable and continuous at 0 function f is completely monotone on $(0, +\infty)$ if and only if

$$f(x) = \int_0^{+\infty} e^{-xt}\, d\alpha(t), \tag{12}$$

where $t \mapsto \alpha(t)$ is bounded and nondecreasing and the integral converges for $0 \leq x < +\infty$.

Suppose that the Maclaurin's series for f converges to $f(x)$ for $x \in (0, R)$, $R > 0$ and let

$$I_n(x) = \sum_{k=n+1}^{+\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

Then by the integral representation we have that

$$(-1)^{n+1} I_n(x) = \frac{1}{n!} \int_0^x (x-t)^n f^{(n+1)}(t)\, dt > 0, \qquad n = 0, 1, \ldots$$

which implies that (ii) and (iii) of Lemma 1 hold for $x \in (0, R)$. Moreover, the following partial generalization of Theorem 1 holds:

THEOREM 3. *If f is a completely monotone function on $(0, +\infty)$ such that its Maclaurin series converges to $f(x)$ for $x \in (0, R)$, then*

$$\frac{I_{n-1}(x) I_{n+1}(x)}{I_n^2(x)} \geq \frac{n}{n+1} \tag{13}$$

for each $x \in (0, R)$ and $n = 0, 1, 2, \ldots$.

Proof. It is easy to see that (13) is equivalent to the statement that the mapping $n \to (-1)^{n+1} n! I_n(x)$ is log-convex for each $x \in (0, R)$. Let $I_n(t, x)$ denote the residual after the nth term in Maclaurin's expansion of the function $x \mapsto e^{-xt}$ for a $t \in (0, +\infty)$. Then by (12) and the uniqueness theorem for Maclaurin's expansion, we have that

$$I_n(x) = \int_0^{+\infty} I_n(t, x)\, d\alpha(t). \tag{14}$$

Now from Theorem 1 it follows that the mapping $n \mapsto (-1)^{n+1} n! I_n(t, x)$ is log-convex for each $x, t \in (0, +\infty)$ and so is the mapping

$$n \mapsto (-1)^{n+1} n! \sum_{i=1}^{k} C_i I_n(t_i, x),$$

where C_i, t_i are arbitrary positive numbers. By passing to a limit we conclude that the mapping

$$n \mapsto (-1)^{n+1} n! \int_0^{+\infty} I_n(t, x)\, d\alpha(t)$$

is also log convex, and (13) follows. ∎

The right inequality in (4) of Theorem 2 cannot be extended to all completely monotone functions. As an example, take $f(x) = 1/(1+x)$, $-1 < x < 1$. Here we have $|I_n| = |x|^{n+1}/(1+x)$, $I_{n-1} I_{n+1} / I_n^2 = 1$ and so the right inequality in (4) does not hold.

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