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On Weak Convergence of Measures on Hilbert Spaces

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The concept of weak convergence of measures on topological spaces depends on a topology of the space. In this paper we discuss the relationship between weak convergence of measures on a separable Hilbert space H equipped with w-* and strong topology. Starting from this point we rederive a well-known condition for the weak convergence on H, hoping to shed a new light on this matter. © 1989 Academic Press, Inc.

1. DIFFERENT CONCEPTS OF WEAK CONVERGENCE

Let H be a real separable Hilbert space. Let $\|\cdot\|$ be the inner product norm and let $\{e_i\}_{i=1}^{\infty}$ be an orthogonal basis with respect to $\|\cdot\|$. We shall consider two topologies on H: the strong (norm) topology and the w-* (weak-star, or, what is the same here, weak) topology. Accordingly, two σ-fields can be defined on H: $\mathscr{B}_s$—the σ-field generated by norm open sets—and $\mathscr{B}_w$—the σ-field generated by w-* open sets. It is a well-known fact that $\mathscr{B}_s = \mathscr{B}_w$. So, the concept of measure is not different in these two topologies. But the concept of convergence is. For convenience, let us recall the following

DEFINITION 1. Let S be a topological space and μ, μ_n $(n = 1, 2, ...)$ measures defined on Borel σ-field of S. We say that μ_n weakly converges to μ $(\mu_n \Rightarrow \mu)$ if for every bounded continuous function $f: S \mapsto R$, the following holds:

$$\int_S f(x)\, d\mu_n(x) \to \int_S f(x)\, d\mu(x), \qquad \text{as} \quad n \to \infty. \tag{1}$$

DEFINITION 2. We say that a set M of measures on S is (weakly sequentially) relatively compact if every sequence of measures in M

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contains a further subsequence which is weakly convergent in the sense of Definition 1.

In what follows, H_s will denote H with the strong (norm) topology and H_w will mean H with the w-* topology. It is clear that $e_n \to 0$ in H_w, but e_n does not converge in H_s. Therefore, if μ_n is a unit mass at e_n, then μ_n weakly converges in H_w, but not in H_s.

2. Some Results on Weak Compactness in H

Theorem 1. *Let $\{\mu_n\}$ be a sequence of positive measures on H, $\sup_n \mu_n(H) = M < \infty$. If*

$$\mu_n \Rightarrow \mu \qquad \text{in} \quad H_w \tag{2}$$

and

$$\lim_N \sup_n \mu_n\left(\sum_{i=N}^{\infty} \langle x, e_i\rangle^2 \geqq \varepsilon\right) = 0, \text{ for all } \varepsilon > 0, \tag{3}$$

then

$$\mu_n \Rightarrow \mu \qquad \text{in} \quad H_s. \tag{4}$$

Proof. By Theorem 2.1. in Billingsley [1], we need to show that the relation (1) holds for every uniformly norm continuous and bounded function $f: H_s \mapsto R$. Let us define $g_N(x) = g_N(\sum_{i=1}^{\infty} \langle x, e\rangle e_i) = \sum_{i=1}^{N} \langle x, e_i\rangle e_i$. Then $\|g_N(x) - g_N(y)\| = \|\sum_{i=1}^{N} \langle x - y, e_i\rangle e_i\| \leqq N\|x - y\|$, so g is a uniformly continuous function for every N. Let f be any uniformly norm continuous real valued function of H_s, $\sup_{x\in H} |f(x)| = M_f$. Define $d_N(x) = f(x) - f(g_N(x))$. Let $\delta > 0$ be fixed. By uniform continuity of f it is possible to find $\varepsilon > 0$ so that $\|x - g_N(x)\|^2 < \varepsilon$ implies $|d_N(x)| < \delta$, for every integer $N \geqq 1$. By condition (3), for such an ε, one can find N_0 so that for every $N \geqq N_0$ we have

$$\mu_n\left\{\sum_{i=N}^{\infty} \langle x, e_i\rangle^2 \geqq \varepsilon\right\} < \delta, \qquad \text{for every } n.$$

So, assume that $N \geqq N_0$ is given. Then we have

$$\begin{aligned}
&\left|\int f(x)\, d\mu_n(x) - \int f(g_N(x))\, d\mu_n(x)\right| = \left|\int d_N(x)\, d\mu_n(x)\right| \\
&\quad \leqq \left|\int_{x:\, \sum_{i=N}^{\infty} \langle x, e_i\rangle^2 < \varepsilon} d_N(x)\, d\mu_n(x)\right| + \left|\int_{x:\, \sum_{i=N}^{\infty} \langle x, e_i\rangle^2 \geqq \varepsilon} d_N(x)\, d\mu_n(x)\right| \\
&\quad < \delta M + 2M_f\delta = \delta(M + 2M_f),
\end{aligned} \tag{5}$$

and this holds for all n.

By continuity of f, the fact that $g_N(x) \to x$ as $N \to \infty$ and the dominated convergence theorem, for $N \geqq N_1$ we have

$$\left| \int f(g_N(x))\, d\mu(x) - \int f(x)\, d\mu(x) \right| < \delta. \tag{6}$$

Since $f(g(\cdot))$ is a continuous bounded function on H_w, by assumption (2) we have that for $n \geqq n_0$.

$$\left| \int f(g_N(x))\, d\mu_n(x) - \int f(g_N(x))\, d\mu(x) \right| < \delta. \tag{7}$$

where $N \geqq \max(N_0, N_1)$ is fixed. Finally, combining (5), (6), and (7) we get $|\int f(x)\, d\mu_n(x) - \int f(x)\, d\mu(x)| \leqq \delta(2 + M + M_f)$, which gives the desired result.

THEOREM 2. *Let μ_n be a sequence of positive measures on H, $\sup_n \mu_n = M < \infty$. Then* (4) *implies* (3).

Proof. For every integer $N > 1$, the set

$$A_N = \left\{ x \colon \sum_{i=N}^{\infty} \langle x, e_i \rangle^2 \geqq \varepsilon \right\}$$

is closed in the norm topology and

$$A_1 \supset A_2 \cdots; \qquad \bigcap_{N=1}^{\infty} A_N = \varnothing. \tag{8}$$

Assume (4). Then by Theorem 2.1 in [1] we have

$$\overline{\lim_n}\, \mu_n(A_N) \leqq \mu(A_N). \tag{9}$$

For $\varepsilon_1 > 0$ arbitrary, but fixed, let N_0 be such that $\mu(A_{N_0}) \leqq \delta_1/2$ (it is possible by (8)). Then (9) gives

$$\overline{\lim_n}\, \mu_n(A_{N_0}) \leqq \varepsilon_1/2. \tag{10}$$

So, there are only finitely many measures, say $\mu_{n_1}, \mu_{n_2}, \ldots, \mu_{n_k}$ such that $\mu_{n_i}(A_{N_0}) \geqq \varepsilon_1$. By (8) there is an integer $N_1 > N_0$ such that $\mu_{n_i}(A_{N_1}) < \varepsilon_1$ for $i = 1, \ldots, k$. Then again by (8) we have $\sup_n \mu_n(A_N) \leqq \varepsilon_1$ for all $N \geqq N_1$, which proves (3).

THEOREM 3. *Let $\mathscr{M}$ be a set of positive measures on H such that* $\sup_{\mu \in \mathscr{M}} \mu(H) = M < \infty$. *Suppose that $\mathscr{M}$ is relatively compact in H_w. Then $\mathscr{M}$ is relatively compact in H_s if and only if*

$$\lim_N \sup_{\mu \in \mathscr{M}} \mu\left(\sum_{i=N}^{\infty} \langle x, e_i \rangle^2 \geqq \varepsilon\right) = 0, \qquad \textit{for every } \varepsilon > 0. \tag{11}$$

Proof. Let $\mathscr{M}$ be a relatively compact set of positive finite measures in H_w. Let μ_n be a sequence in $\mathscr{M}$. Then there is a weakly convergent in H_w subsequence $\mu_{n'} \Rightarrow \mu$ for some μ. Then by Theorem 1, (11) implies weak convergence of μ_n in H_s.

Conversely, suppose that $\mathscr{M}$ is relatively compact in H_s and (11) does not hold. Then there is an $\varepsilon > 0$, and a $\lambda > 0$, such that for every n there exist an $N \geqq n$ and a $\mu_n \in \mathscr{M}$, so that

$$\mu_n \left\{ \sum_{i=N}^{\infty} \langle x, e_i \rangle^2 \geqq \varepsilon \right\} > \lambda. \tag{12}$$

The sequence μ_n has a weakly convergent subsequence in H_s, which, together with (12), contradicts Theorem 2. Therefore, (11) holds.

3. CONNECTION WITH KNOWN RESULTS

Let μ be a positive finite measure on H. The characteristic functional of μ is defined by $f(x) = \int \exp(i\langle x, y \rangle)\, d\mu(y)$. It is well known (see [4]) that f is continuous in so-called I-topology, determined by Hilbertian semi-norms (HSN) p with the properties $\sum_{i=1}^{\infty} p^2(e_i) < \infty$, $p(x) \leqq C\|x\|$, where C is a constant that may vary with p. If a HSN p satisfies the above conditions, we say that $p \in I$. Corresponding inner products $p(\cdot, \cdot)$ are defined in the usual manner through $p(\cdot)$. For details on this, see [2–4]. It follows from the continuity of f that for every $\varepsilon > 0$ there is a HSN $p_{\mu,\varepsilon} \in I$ (not unique!) such that for every $x \in H$:

$$\mu(H) - \operatorname{Re} f(x) \leqq p_{\mu,\varepsilon}^2(x) + \varepsilon. \tag{13}$$

In what follows, $p_{\mu,\varepsilon}$ will always have the meaning described here. $\mathscr{M}$ will denote a set of positive measures on H, with $\sup_{\mu \in \mathscr{M}} \mu(H) = M < \infty$. $\{p_{\mu,\varepsilon}\}_{\mu \in \mathscr{M}}$ is a set of HSN that, for given $\varepsilon > 0$, corresponds to $\mu \in \mathscr{M}$ in the sense of (13). B_r will denote a closed ball in H_s with radius r, i.e., $B_r = \{x \in H\colon \|x\| \leqq r\}$. If $\{e_i\}$ is an orthonormal basis in H, then for $x \in H$, $x_i = \langle x, e_i \rangle$.

THEOREM 4. *$\mathscr{M}$ is relatively compact in H_w if and only if for every $\varepsilon > 0$ there is a set $\{p_{\mu,\varepsilon}\}_{\mu \in \mathscr{M}}$, such that* $\sup_{\mu \in \mathscr{M}} \sum_{i=1}^{\infty} p_{\mu,\varepsilon}^2(e_i) < \infty$. (14)

Proof. Let us first note that Prohorov's theorem holds in H_w. Namely, since H_w is the countable union of metrizable balls B_r $(r=1, 2, ...)$, the assertion follows from Prohorov's theorem in metric spaces and a standard diagonalization argument (see also [5]). By Prohorov's theorem, the relative compactness of $\mathscr{M}$ in H_w is equivalent to:

$$\begin{aligned}&\text{for every } \varepsilon>0 \text{ there is an } r>0 \text{ such that}\\&\text{for every } \mu\in\mathscr{M},\ \mu(B_r^c)\leqq\varepsilon.\end{aligned}\tag{15}$$

We shall show that (14) is equivalent to (15):

(i) Suppose that (14) holds. Let $\{p_{\mu,\varepsilon}\}$ be as in (14). Let $\varepsilon>0$ be fixed and let $N>0$ be a fixed integer. Let $A_{r,N}=\{y: \sum_{i=1}^N \langle y, e_i\rangle^2>r^2\}$. For a $y\in A_{r,N}$, we have $1-\exp(-\sum_{i=1}^N \langle y, e_i\rangle^2/2r^2)>1-\exp(-\frac{1}{2})>\frac{1}{3}$. So, for every $\mu\in\mathscr{M}$, the following holds:

$$\begin{aligned}\mu(A_{r,N})/3&<\int_{A_{r,N}}\left(1-\exp\left(-\sum_{i=1}^N y_i^2/2r^2\right)\right)d\mu(y)\\&<\mu(H)-\int\exp\left(-\sum_{i=1}^N y_i^2/2r^2\right)d\mu(y).\end{aligned}\tag{16}$$

Let $\mathscr{G}$ be a Gaussian measure on R^N: $d\mathscr{G}(y)=\prod_{i=1}^N \mathscr{N}_{r,i}(dy_i)$, where $\mathscr{N}_{r,i}$ is Gaussian $\mathscr{N}(0, 1/r^2)$. Then $\exp(-\sum_{i=1}^N y_i^2/2r^2)$ is the characteristic functional of $\mathscr{G}$; so we have

$$\begin{aligned}&\int\exp\left(-\sum_{i=1}^N y_i^2/2r^2\right)d\mu(y)\\&\qquad=\int_{R^N}\int_H\exp\left(i\sum_{i=1}^N y_i x_i\right)d\mu(y)\,d\mathscr{G}(x).\end{aligned}\tag{17}$$

Let $f_N(x)=\int\exp(i\sum_{i=1}^N y_i x_i)\,d\mu(y)$. If $x=\sum_{i=1}^N x_i e_i$, then $f_N(x)=f(x)$; so by (13) we have $\operatorname{Re} f_N(x)\geqq\mu(H)-p_{\mu,\varepsilon}^2(x)-\varepsilon$. Then from (17) it follows:

$$\begin{aligned}\exp\left(-\sum_{i=1}^N y_i^2/2r^2\right)d\mu(y)&\geqq\mu(H)-\int_H p_{\mu,\varepsilon}^2(x)\,d\mathscr{G}(x)-\varepsilon\\&=\mu(H)-\int_H\sum_{i,j=1}^N x_i x_j p_{\mu,\varepsilon}(e_i, e_j)\,d\mathscr{G}(x)-\varepsilon\\&=\mu(H)-(1/r^2)\sum_{i=1}^N p_{\mu,\varepsilon}^2(e)-\varepsilon.\end{aligned}\tag{18}$$

Now (16) and (18) imply

$$\mu\left\{y: \sum_{i=1}^{N} \langle y, e_i\rangle^2 > r^2\right\} < 3\left(\sum_{i=1}^{N} p_{\mu,\varepsilon}^2(e_i)\right)\Big/ r^2 + \varepsilon.$$

Letting $N \to \infty$ and taking supremum over all $\mu \in \mathcal{M}$, we obtain

$$\sup_{\mu} \mu(B_r^c) \leqq 3 \sup_{\mu}\left(\sum_{i=1}^{\infty} p_{\mu,\varepsilon}^2(e_i)\right)\Big/ r^2 + 3\varepsilon,$$

so (15) holds.

(ii) Suppose now that (15) holds. For $\varepsilon > 0$ fixed, let B_r be the corresponding ball in the sense of (15). Let $f_\mu(\cdot)$ be the characteristic functional of $\mu \in \mathcal{M}$. Then (15) implies

$$\begin{aligned} \mu(H) - \operatorname{Re} f_\mu(y) &= \int (1 - \cos\langle x, y\rangle)\, d\mu(x) \\ &\leqq \int_{B_r} (1 - \cos\langle x, y\rangle)\, d\mu(x)) + 2\varepsilon \\ &\leqq (\tfrac{1}{2}) \int_{B_r} \langle x, y\rangle^2\, d\mu(x) + 2\varepsilon. \end{aligned}$$

Define $p_{\mu,\varepsilon}$ by

$$p_{\mu,\varepsilon}^2(y) = (\tfrac{1}{2}) \int_{B_r} \langle x, y\rangle^2\, d\mu(x). \tag{19}$$

It is not hard to see that $p_{\mu,\varepsilon}$ is a Hilbertian semi-norm $p_{\mu,\varepsilon}(y) \leqq r\mu(H)\, \|y\|/\sqrt{2}$. Moreover,

$$\sum_{i=1}^{\infty} p_{\mu,\varepsilon}^2(e_i) = \tfrac{1}{2} \int_{B_r} \|x\|^2\, d\mu(x) < r^2 M/2,$$

so $p_{\mu,\varepsilon} \in I$ and (14) holds.

THEOREM 5. *Suppose that $\mathcal{M}$ is relatively compact in H_w. Then* (20) *and* (21) *below are equivalent:*

For every $\varepsilon > 0$, there is a $\{p_{\mu,\varepsilon}\}_{\mu \in \mathcal{M}}$ such that $\lim_N \sup_{\mu \in \mathcal{M}} \sum_{i=N}^{\infty} p_{\mu,\varepsilon}^2(e_i) = 0.$ (20)

For every $\varepsilon > 0$, $\lim_N \sup_{\mu \in \mathcal{M}} \mu(\sum_{i=N}^{\infty} \langle x, e_i\rangle^2 > \varepsilon) = 0.$ (21)

Proof. Assume that (20) holds. Relation (13) implies

$$\operatorname{Re} \int \exp(i\langle x, y\rangle)\, d\mu(x) \geqq \mu(H) - \varepsilon - p^2_{\mu,\varepsilon}(y).$$

Letting $y = \sum_{j=N}^{S} a_j e_j$ (a_j real numbers, S and N integers, $S \geqq N > 0$) and integrating with respect to $\prod_{i=N}^{S} \mathcal{N}_i(da_i)$, where $\mathcal{N}_i \sim \mathcal{N}(0, 1)$ are i.i.d., we obtain

$$\int \exp\left(-\tfrac{1}{2} \sum_{j=N}^{S} \langle x, e_j\rangle^2\right) d\mu(x) \geqq \mu(H) - \varepsilon - \sum_{j=N}^{S} p^2_{\mu,\varepsilon}(e_j). \tag{22}$$

Relation (22) holds for every $\varepsilon > 0$, $S \geqq N > 0$, $\mu \in \mathcal{M}$. Letting $S \to \infty$ and using the monotone convergence theorem, we have

$$\int \exp\left(-\tfrac{1}{2} \sum_{j=N}^{\infty} \langle x, e_j\rangle^2\right) d\mu(x) \geqq \mu(H) - \varepsilon - \sum_{j=N}^{\infty} p^2_{\mu,\varepsilon}(e_j). \tag{23}$$

Let $\sum_{j=N}^{\infty} p^2_{\mu,\varepsilon}(e_j) = S^{\mu}_N(\varepsilon)$; $\frac{1}{2}\sum_{j=N}^{\infty} \langle x, e_j\rangle^2 = X(N)$. Then from (23) it follows that $\mu(H) - \varepsilon - S^{\mu}_N(\varepsilon) \leqq \int \exp(-X(N))\, d\mu(x) < (1 - \exp(-\lambda))\, \mu(X(N) \leqq \lambda) + \exp(-\lambda)\, \mu(H)$, for every N, μ, $\varepsilon > 0$ and $\lambda > 0$. Thus,

$$\sup_{\mu \in \mathcal{M}} \mu(X(N) > \lambda) < (\varepsilon + \sup_{\mu \in \mathcal{M}} S^{\mu}_N(\varepsilon))/(1 - \exp(-\lambda)).$$

Letting $N \to \infty$ and then $\varepsilon \to 0$ we obtain (21) (with λ in place of ε). Note that in this part of the proof we did not use the assumption that $\mathcal{M}$ is relatively compact in H_w.

Conversely, assume that $\mathcal{M}$ is relatively compact in H_w and that (21) holds. As we have shown in the proof of Theorem 4, $p_{\mu,\varepsilon}$ defined by (19) is a seminorm satisfying (13). Further,

$$2 \sum_{i=N}^{\infty} p^2_{\mu,\varepsilon}(e_i) = \int_{B_r} \sum_{i=N}^{\infty} \langle x, e_i\rangle^2\, d\mu(x) < \lambda\mu(H) + r^2 \mu\left(\sum_{i=N}^{\infty} \langle x, e_i\rangle^2 > \lambda\right).$$

Taking the supremum over $\mu \in \mathcal{M}$, letting $N \to \infty$, and, finally $\lambda \to 0$, we obtain (20).

Remark 1. If (14) holds for some choice of $\{p_{\mu,\varepsilon}\}_{\mu\in\mathscr{M}}$, it need not hold for every choice. For example, let $\mathscr{M}$ be a sequence of measures, $\{p_{\mu_n,\varepsilon}\}$ a choice of seminorms for which (13) and (14) hold. Denote $p_{\mu_n,\varepsilon}$ by p_n, and let $S_n=\sum_{i=1}^{\infty}p_n^2(e_i)$. If $q^2(x)=np^2(x)/S_n$, then $\{q_n\}$ is a choice of Hilbertian seminorms for which (13) holds, but not (14). A similar example can be found for (21).

Remark 2. Relations (14) and (21) are well-known necessary and sufficient conditions for the relative compactness in H_s (see Parthasarathy [3]). We have proved here that (14) is the condition for relative compactness in H_w, whereas (21) is the additional requirement in H_s. The separate condition in H_s is useful in cases when we do not need stronger convergence, for example, when we want to establish the existence of a particular measure by construction of a sequence of measures that converges to it.

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