
**Krull’s theory for the Double Gamma function**

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Dedicated to Professor H. M. Srivastava on the Occasion of his Seventieth Birth Anniversary

The Barnes’ $G$-function $G(x) = 1/\Gamma_2$, satisfies the functional equation $\log G(x + 1) - \log G(x) = \log \Gamma(x)$. We complement W. Krull’s work in *Bemerkungen zur Differenzengleichung* $g(x + 1) - g(x) = \varphi(x)$, Math. Nachrichten 1 (1948), 365-376 with additional results that yield a different characterization of the function $G$, new expansions and sharp bounds for $G$ on $x > 0$ in terms of Gamma and Digamma functions, a new expansion for the Gamma function and summation formulae with Polygamma functions.

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1. Introduction

The functional relation

$$g(x + 1) - g(x) = f(x), \quad x > a$$

appears naturally in the theory of the Gamma function, with $g(x) = \log \Gamma(x)$ and $f(x) = \log x$, as well with numerous related functions as Polygamma or Multiple Gamma functions in place of $g$, and with appropriate choice of $f$. The theory of these functional equations, with $x$ being a real or a complex number, is studied in Chapter 2 of Campbell’s book [1], in the article [5] by Dufresnoy and Pisot and in Ruijsenaars’ papers [12] and [13] with reference to Gamma and related functions, and assuming that $f$ is an analytic function of a complex variable, as it usually is in applications. However, the pioneering work by Krull [7] on this type of equations is not cited in any of mentioned references, and seems to be very little known. The main feature of Krull’s approach is that it uses convexity as virtually the only tool, and produces expansions for solutions $g$ of (1), where $f$ is a given convex or concave function or a sum of a convex and a concave function with

$$\lim_{x \to +\infty} (f(x + 1) - f(x)) = 0.$$

In this paper we revisit Krull’s theory and provide several new results that enable applications to Barnes’ function $G(z) = 1/\Gamma_2(z)$, where $\Gamma_2(z)$ is so called Double Gamma function. In last several decades this function has been studied in a number of interesting papers devoted to topics as its role in a study of determinants of Laplacians in [14], results connecting $G$-function to other special functions, sums and series as, for example, in [2], [3], [4], asymptotic expansions as in [6] and references therein, and results in already mentioned papers [5],[12] and [13]. The $G$-function is an entire function of a complex variable and $\log G(x)$ satisfies (1) with $f(x) = \log \Gamma(x)$, hence the condition (2) does not hold. With a goal to expand the domain of Krull’s theory, we present results regarding uniform convergence (corresponding parts in Lemmas 2–4 and Theorems 1 and 2) and expansions in the form of double inequalities (Theorems 1a and 2a). Sections 2 and 3 are devoted to Krull’s theory in its full generality, and Section 4 presents applications to the Barnes $G$-function. We give a new characterization theorem for the function $G$, two new expansions and the corresponding sequence of upper and lower bounds for the function $G$ of a positive real argument, in terms of Gamma and Digamma functions, a new expansion for the Gamma function and summation formulae with Polygamma functions.
2. SOME AUXILIARY RESULTS

In derivation of results in this section, we repeatedly make use of the following equivalent definition of convexity (see, for example \[8, 16.B.3.a\]): A function \(f\) defined on an arbitrary interval \(I\) is convex on \(I\) if and only if

\[
\frac{f(y_2) - f(x_2)}{y_2 - x_2} \leq \frac{f(y_1) - f(x_1)}{y_1 - x_1} \quad \text{whenever } x_1 < y_1 \leq y_2 \text{ and } x_1 \leq x_2 < y_2.
\]

(3)

Although in this paper we apply Krull’s theory to differentiable functions, in order to retain the full generality we do not want to assume differentiability in most of results that are presented here.

Given \(a \in \mathbb{R}\), let \(C_a\) be the class of real valued functions that are convex on \((a, +\infty)\) and satisfy condition (2). In the next lemma we collect some interesting properties of functions in \(C_a\). Proofs of all statements can be easily obtained using (3).

**Lemma 1.** Suppose that \(f \in C_a\). Then the following holds:

(i) The function \(f\) is decreasing on \((a, +\infty)\) or it is constant on \((a, +\infty)\).

(ii) The difference \(f(x + d) - f(x)\) is non-decreasing with respect to \(x > a\), for any fixed \(d > 0\) and non-increasing with respect to \(x > a - d\) if \(d < 0\).

(iii) For any \(d \in \mathbb{R}\),

\[
\lim_{x \to +\infty} (f(x + d) - f(x)) = 0, \quad \text{for every } d \in \mathbb{R}.
\]

(iv) One sided derivatives \(f'_+(x)\) and \(f'_-(x)\) converge to zero as \(x \to +\infty\).

**Lemma 2.** Let \(f\) be a non-constant function in \(C_a\). Then for every \(x_0 > a\) the series

\[
\sum_{k=0}^{+\infty} \left(\frac{f(x + k) - f(x_0 + k)}{x - x_0} - (f(x_0 + k + 1) - f(x_0 + k))\right) := \sum_{n=0}^{+\infty} (-1)^{n+1} b_n(x),
\]

where, for \(k = 0, 1, \ldots\)

\[
b_{2k}(x) = \frac{f(x_0 + k) - f(x + k)}{x - x_0}, \quad b_{2k+1}(x) = f(x_0 + k) - f(x_0 + k + 1),
\]

is of Leibniz type and uniformly convergent on \([\xi, x_0 + 1]\setminus\{x_0\}\), for any \(\xi\) such that \(\xi > a\) and \(x_0 - 1 \leq \xi < x_0 + 1\).

**Proof.** First, note that by Lemma 1(i), \(f\) is decreasing, so \(b_n(x) > 0\) for all \(n\) and \(x \neq x_0\). By Lemma 1(iii)

\[
\lim_{k \to +\infty} b_{2k}(x) = \lim_{k \to +\infty} b_{2k+1}(x) = 0
\]

for all \(x > a\). Further, by convexity of \(f\) we have, for \(x \in (x_0, x_0 + 1]\),

\[
\frac{f(x + k) - f(x_0 + k)}{x - x_0} \leq f(x_0 + k + 1) - f(x_0 + k) \leq \frac{f(x + k + 1) - f(x_0 + k + 1)}{x - x_0},
\]

and for \(x \in [\xi, x_0]\)

\[
\frac{f(x_0 + k) - f(x + k)}{x_0 - x} \leq f(x_0 + k + 1) - f(x_0 + k) \leq \frac{f(x_0 + k + 1) - f(x + k + 1)}{x_0 - x},
\]

which in both cases yields \(b_{2k} \geq b_{2k+1} \geq b_{2k+2}\), so the sequence \(\{b_n(x)\}\) is positive, non-increasing, and converges to zero.

Let \(R_n(x)\) be the residual after \(n\)-th term in (5). By convexity \(b_{2k}(x) \leq b_{2k}(\xi)\) for \(x \geq \xi\) and so

\[
|R_{2k-1}(x)| \leq \frac{f(x_0 + k) - f(\xi + k)}{\xi - x_0}, \quad |R_{2k}| \leq f(x_0 + k) - f(x_0 + k + 1),
\]

(6)

hence \(R_n(x) \to 0\) as \(n \to +\infty\), uniformly on \(x\).
Lemma 3. Let $f$ be a non-constant function with $f \in C_a$ or $-f \in C_a$. For integers $k \geq 0$, define $a_{2k}(x) = f(x + k) - f(x_0 + k)$, $a_{2k+1}(x) = (x - x_0)(f(x_0 + k + 1) - f(x_0 + k))$ and let

$$S(x) = \sum_{n=0}^{+\infty} (-1)^n a_n(x)$$

Then the series in (7) is of Leibniz type and uniformly convergent on $x \in [\xi, x_0 + 1]$, for any $\xi$ such that $\xi > a$ and $x_0 - 1 \leq \xi < x_0 + 1$. Moreover, for a convex function $f$ we have, for $n = 1, 2, \ldots$

$$a_n(x) > 0 \text{ for } x < x_0, \text{ and } a_n(x) < 0 \text{ for } x > x_0.$$

If $f$ is concave, then inequalities in (8) are reversed.

Proof. Let $A_n(x)$ be the $n$-th partial sum of the series in (7) and denote by $B_n(x)$ the partial sum of the series in (5) of Lemma 2. Noticing that $A_n(x) = (x - x_0)B_n(x)$, and $|x - x_0| \leq 1$ for $x \in (\xi, x_0 + 1)$, the statements follow from Lemma 2.

Lemma 4. With the setup as in Lemma 3, the series

$$S(x) = \sum_{k=0}^{+\infty} (a_{2k}(x) - a_{2k+1}(x))$$

is convergent for all $x > a$ and uniformly convergent on any interval $[c, d]$, where $a < c < d$.

Proof. We will prove the lemma for a convex function $f$, as the concave case can be treated similarly. Let $S_n(x)$ be the $n$-th partial sum of the series (9). Then for $x > a$,

$$S_n(x + 1) = S_n(x) + f(x + n + 1) - f(x_0 + n + 1) + f(x_0) - f(x).$$

By Lemma 1, $f(x + n + 1) - f(x_0 + n + 1) \to 0$ as $n \to +\infty$ and so $S_n(x + 1)$ is convergent if and only if $S_n(x)$ is, provided that $x > a$. Since by Lemma 3, $S_n(x)$ converges in $[x_0, x_0 + 1]$, it follows that it converges for all $x > a$.

From (10) it follows that $S(x + 1) = S(x) + f(x_0) - f(x)$, and so, with $R_n(x) = S(x) - S_n(x)$ we have

$$R_n(x + 1) - R_n(x) = f(x_0 + n + 1) - f(x + n + 1).$$

To find a bound for the right hand side in (11), we consider cases $a < x < x_0$ and $x > x_0$ separately. Without loss of generality, we assume that $a < c < x_0 < d$, and that $x \in [c, d]$, $x + 1 \in [c, d]$.

If $a < c \leq x < x_0$, then, by convexity of $f$,

$$0 \geq \frac{f(x_0 + n + 1) - f(x + n + 1)}{x_0 - x} \geq \frac{f(x_0 + n + 1) - (c + n + 1)}{x_0 - c},$$

and so

$$|R_n(x + 1) - R_n(x)| \leq \frac{x_0 - x}{x_0 - c} (f(c + n + 1) - f(x_0 + n + 1)) \leq f(c + n + 1) - f(x_0 + n + 1) := A_n.$$

Note that $A_n$ does not depend on $x$ and converges to zero as $n \to +\infty$.

If $x_0 < x \leq d$, then, again by convexity of $f$, we have that

$$\frac{f(x + n + 1) - f(x_0 + n + 1)}{x - x_0} \geq \frac{f(x + \varepsilon + n + 1) - f(x_0 + n + 1)}{\varepsilon},$$

for any $\varepsilon > 0$ such that $x \geq x_0 + \varepsilon$. By letting $\varepsilon \to 0_+$, we see that

$$0 \geq \frac{f(x + n + 1) - f(x_0 + n + 1)}{x - x_0} \geq f'_+(x_0 + n + 1),$$

and finally

$$|R_n(x + 1) - R_n(x)| \leq (x - x_0)|f'_+(x_0 + n + 1)| \leq (d - x_0)|f'_+(x_0 + n + 1)| := B_n,$$

where $B_n$ does not depend on $x$ and converges to zero as $n \to +\infty$ by Lemma 1 (iv).

Now let $C_n = \max(A_n, B_n)$. Then

$$R_n(x) - C_n \leq R_n(x + 1) \leq R_n(x) + C_n \quad \text{and} \quad R_n(x + 1) - C_n \leq R_n(x) \leq R_n(x + 1) + C_n.$$

(12) $R_n(x) - C_n \leq R_n(x + 1) \leq R_n(x) + C_n$ and $R_n(x + 1) - C_n \leq R_n(x) \leq R_n(x + 1) + C_n$. \end{document}
for all $x$ such that both $x$ and $x + 1$ are in $[c, d]$. Since in Lemma 3 we already proved uniform convergence for $x \in I$, where $I = [\xi, x_0 + 1]$, with $\xi \geq x_0 - 1$, we see that the uniformity can be shown to hold to the right or to the left of $I$, by repeating (12) if necessary a finite number of times, till we reach given boundaries $c$ and $d$.

**Lemma 5.** Let $f \in C_a$. Then for every $x > a, h_1, h_2 \in (0, 1)$ such that $x - h_1 > a$, the series

$$
\sum_{k=0}^{+\infty} \left( \frac{f(x + k + h_2) - f(x + k)}{h_2} - \frac{f(x + k) - f(x + k - h_1)}{h_1} \right)
$$

converges and its sum $S(x)$ satisfies

$$
0 \leq S(x) \leq \frac{f(x - h_1) - f(x)}{h_1}, \quad \lim_{x \to +\infty} S(x) = 0.
$$

Moreover, the series (13) is uniformly convergent on $x \in [c, +\infty)$ for any $c > a$.

**Proof.** By convexity, all terms of the series in (13) are non-negative. Also by convexity,

$$
\frac{f(x + k + h_2) - f(x + k)}{h_2} \leq \frac{f(x + k + 1) - f(x + k + 1 - h_1)}{h_1}.
$$

Denoting by $D_{k+1}$ the right hand side above and by $S_n$ the $n$-th partial sum of the series (13), we have that

$$
0 \leq S_n \leq \sum_{k=0}^{n} (D_{k+1} - D_k) = D_{n+1} - D_0 = \frac{f(x + n + 1) - f(x + n + 1 - h_1)}{h_1} - \frac{f(x) - f(x - h_1)}{h_1}.
$$

Now, by Lemma 1(i), $f(x + n + 1) \leq f(x + n + 1 - h_1)$ and $f(x) \leq f(x - h_1)$ for $x - h_1 > a$, so

$$
0 \leq S_n \leq \frac{f(x - h_1) - f(x)}{h_1}
$$

and it follows that the series is convergent and that its sum $S(x)$ satisfies (14).

The uniform convergence on $[c, +\infty)$ can be shown in the same way as in Lemma 2.

3. Main results

In what follows we will make use of the following two conditions.

**Condition A.** We say that $f$ satisfies condition A if $f(x) = f_1(x) - f_2(x)$, where $f_1, f_2 \in C_a$.

**Condition B.** A function $g$ satisfies condition B if for any $h_1, h_2$ such that $0 < h_1 < \delta$ for some $\delta > 0$, we have that

$$
\lim_{x \to +\infty} \left( \frac{g(x + h_2) - g(x)}{h_2} - \frac{g(x) - g(x - h_1)}{h_1} \right) = 0.
$$

Note that the condition B is satisfied with any twice differentiable function $g$ such that $g''(x) \to 0$ as $x \to +\infty$.

In order to accommodate the statements for use in applications in the next section, the formulae that depend on convexity of $f$ and $g$ are here expressed for the case of concave $f$ and convex $g$.

**Lemma 6.** Suppose that the equation (1) has a solution $g$ that satisfies condition B. Then all solutions that satisfy condition B are of the form $g + C$, where $C$ is an arbitrary constant.

**Proof.** If $g_1$ is another solution of (1) that satisfies condition B, then the difference $\rho(x) = g_1(x) - g(x)$ also satisfies condition B and $\rho(x + 1) = \rho(x)$ for every $x > a$. Then it can be proved that $\rho$ is affine on $x > a$, and by periodicity it is a constant, see details in [11, Theorem 6.3].

**Lemma 7.** Suppose that $f$ satisfies the condition A, and let $g$ be a convex solution of (1). Then $g$ satisfies the condition B. Moreover, any solution of (1) that satisfies the condition B is of the form $g + C$, where $C$ is an arbitrary constant. In particular, any convex solution of (1) is of the form $g + C$ and any twice differentiable solution with a second derivative converging to zero as $x \to +\infty$ is also of the form $g + C$. 
Proof. Let \( f \) satisfy the condition A, and let \( g \) be a convex solution of (1). Then for any \( h_1, h_2 \in (0, 1) \) we have that
\[
f(x + 1) - f(x) = \frac{(g(x + 2) - g(x + 1)) - (g(x + 1) - g(x))}{h_2} \geq \frac{g(x + 1 + h_2) - g(x + 1) - g(x + 1) - g(x)}{h_1} \geq 0.
\]
By condition A, \( f(x + 1) - f(x) \to 0 \) as \( x \to +\infty \) and so we conclude that \( g \) satisfies the condition B. Then by Lemma 6, all solutions of (1) that satisfy the condition B are of the form \( g \). As we just proved, any convex solution of (1) satisfies the condition B, and so, any convex solution is of the form \( g + C \). Finally, as we remarked before Lemma 6, any function with the second derivative converging to zero as \( x \to +\infty \) satisfies condition B and hence any such a solution is of the form \( g + C \).

**Theorem 1.** Suppose that \( f \) satisfies condition A. For \( x_0 > a \) being fixed, let
\[
g(x) = \int_{x_0}^{x} f(u) \, du - \frac{1}{2} f(x) + \sum_{k=0}^{+\infty} \left( \int_{x+k}^{x+k+1} f(u) \, du - \frac{1}{2} (f(x+k+1) + f(x+k)) \right),
\]
where \( x > a \). Then the series in (16) converges uniformly on \( x \in [c, +\infty) \) for any \( c > a \) and the function \( x \mapsto g(x) \) is well defined on \( (a, +\infty) \). Further,

(i) The function \( g \) is a solution of equation (1) on \( x > a \).
(ii) The function \( g \) satisfies condition B.
(iii) The following relation holds:
\[
g(x) = \int_{x_0}^{x} f(u) \, du - \frac{1}{2} f(x) + o(1), \quad (x \to +\infty).
\]
(iv) In addition, if \( f \) is either convex or concave, then
\[
|S(x)| \leq \frac{1}{2} |f(x+1/2) - f(x)|,
\]
where \( S(x) \) is the sum of the series in (16).

Proof. It is clear that if the assertion of the theorem holds in the case of convex \( f \) satisfying condition A, then it must hold in a general case of condition A. So, let us assume that \( f \in C_a \).

We first have to prove that the series in (16) converges. Let us introduce the notation
\[
D(x) = \int_{x}^{x+1} f(u) \, du - \frac{1}{2} (f(x+1) + f(x)).
\]
Then by Hadamard’s inequalities,
\[
f(x + 1/2) \leq \int_{x}^{x+1} f(u) \, du \leq \frac{1}{2} (f(x) + f(x+1))
\]
and therefore
\[
0 \geq D(x+k) \geq f(x+k+1/2) - \frac{1}{2} (f(x+k+1) + f(x+k)) \geq \frac{1}{4} \left( f(x+k+1) - f(x+k+1/2) - f(x+k+1/2) - f(x+k) \right).
\]

Denote by \( h(x+k) \) the expression on the right hand side of (19). By Lemma 5, applied with \( h_1 = h_2 = 1/2 \) and with \( x \) replaced by \( x+1/2 \), the series \( \sum h(x+k) \) converges for any \( x > a \) and its sum converges to zero as \( x \to +\infty \). By (19), the same holds true for the series \( \sum D(x+k) \). This proves (iii). The statement (iv) follows from (19) and (15).

To prove the uniform convergence of the series, let us observe that \( R_{n-1}(x) = S(x+n) \), where \( R_{n-1}(x) \) is the residual in (16). Then for \( f \) being convex we have the following estimate:
\[
|R_{n-1}(x)| \leq \frac{1}{2} |f\left(x + n + \frac{1}{2}\right) - f(x+n)| \leq \frac{1}{2} |f\left(c + n + \frac{1}{2}\right) - f(c+n)| \to 0,
\]
where the second inequality follows from Lemma 1 (ii). This proves the uniform convergence.
In order to prove (ii), note that by (iii) and Lemma 5, it suffices to show that the condition B is satisfied by the function \( F \) defined by \( F(x) = \int_{x_0}^{x} f(u) \, du \). For any \( h_1, h_2 > 0 \) we have

\[
\frac{F(x + h_2) - F(x)}{h_2} - \frac{F(x) - F(x - h_1)}{h_1} = \frac{1}{h_2} \int_{x}^{x+h_2} f(u) \, du - \frac{1}{h_1} \int_{x-h_1}^{x} f(u) \, du
\]

\[
= f(x) - f(x_1),
\]

where \( x_1 \in (x - h_1, x) \) and \( x_2 \in (x, x + h_2) \) (by the mean value theorem and continuity of a convex function in an open interval). By Lemma 1(i), \( f \) is non-increasing and therefore,

\[
0 \geq f(x_2) - f(x_1) \geq f(x + h_2) - f(x - h_1) \to 0 \quad \text{as} \quad x \to +\infty.
\]

Therefore, the condition B holds for \( g \), and for any \( n \in \mathbb{N} \), it is non-increasing between \( x + n + 1 \) and \( x + n + 2 \), which yields

\[
|g_n(x + 1) - g_n(x) - f(x)| \leq \frac{1}{2} (f(x + n + 1) - f(x + n + 2)) \to 0 \quad \text{as} \quad n \to +\infty,
\]

thus \( g(x + 1) - g(x) = g(x) \).

If \( f \) is an affine function, then the series in Theorem 1 vanishes. In the next theorem we complement Theorem 1 by analysis of the remainder when \( f \) is a strictly concave function.

**Theorem 1a.** Let \( f \) be a strictly concave function that satisfies condition A and let \( g \) be defined as in (16) of Theorem 1. Then for every \( x > a \) we have the following inequalities:

\[
\int_{x_0}^{x} f(u) \, du - \frac{1}{2} f(x) < g(x) < \int_{x_0}^{x} f(u) \, du + \frac{1}{2} \int_{x}^{x+1} f(u) \, du - f(x)
\]

and, for \( n \geq 1,

\[
g(x) > \int_{x_0}^{x} f(u) \, du - \frac{1}{2} f(x) + \frac{1}{2} \sum_{k=0}^{n-1} \left( \int_{x+k}^{x+k+1} f(u) \, du - \frac{1}{2} (f(x+k+1) + f(x+k)) \right)
\]

as well as

\[
g(x) < \int_{x_0}^{x} f(u) \, du - \frac{1}{2} f(x) + \frac{1}{2} \sum_{k=0}^{n-1} \left( \int_{x+k}^{x+k+1} f(u) \, du - \frac{1}{2} (f(x+k+1) + f(x+k)) \right)
\]

\[
+ \frac{1}{2} \left( \int_{x+n}^{x+n+1} f(u) \, du - f(x+n) \right).
\]

For fixed \( x \), the bounds become infinitely sharp as \( n \to +\infty \). In addition, if \( f''(x) \to 0 \) as \( x \to +\infty \), then for a fixed \( n \), the absolute error in (21)–(23) converges to zero as \( x \to +\infty \).

**Proof.** Let

\[
a_k(x) = \frac{1}{2} \left( \int_{x+k}^{x+k+1} f(u) \, du - f(x+k) \right), \quad b_k(x) = \frac{1}{2} \left( f(x+k+1) - \int_{x+k}^{x+k+1} f(u) \, du \right).
\]

Since the series in (16) is convergent, \( g \) can be represented as

\[
g(x) = \int_{x_0}^{x} f(u) \, du - \frac{1}{2} f(x) + a_0(x) - b_0(x) + a_1(x) - b_1(x) + \cdots
\]
As \( g \) is strictly concave, then it is increasing by Lemma 1, and so we find that \( a_k > 0 \) and \( b_k > 0 \). Further, again by concavity we have that

\[
\int_{x+k}^{x+k+1} f(u) \, du > \frac{1}{2} (f(x + k) + f(x + k + 1)),
\]

which yields \( a_k > b_k \). By Hadamard’s inequality for strictly concave functions, we have that

\[
f(x + k + 1) > \frac{1}{2} \int_{x+k}^{x+k+2} f(u) \, du,
\]

and so, \( b_k > a_{k+1} \). Since both \( a_k \) and \( b_k \) converge to zero as \( k \to +\infty \), we have that the series \( a_0 - b_0 + a_1 - b_1 + \cdots \) is of Leibniz type and the conclusion of the theorem follows. The statement about the behavior of bonds for fixed \( n \) as \( x \to +\infty \) follows from the results in [9].

**Theorem 2.** Suppose that \( f \) satisfies condition A. For \( x_0 \geq a \) and \( y_0 \in \mathbb{R} \) being fixed, define a function \( g^* \) by

\[
g^*(x) = y_0 + (x - x_0) f(x_0)
\]

\[
- \sum_{k=0}^{+\infty} \left( f(x + k) - f(x_0 + k) \right) - (x - x_0) \left( f(x_0 + k + 1) - f(x_0 + k) \right).
\]

Then

(i) The function \( g^* \) satisfies (1) on \( x > a \) and \( g^*(x_0) = y_0 \).

(ii) The function \( g^* \) satisfies the condition B.

(iii) The series in (25) converges uniformly in any compact interval in \((a, +\infty)\).

**Proof.** The uniform convergence of the series in (25) is proved in Lemma 4. Part (i) can be verified using relation (10). Condition B follows from Lemma 5.

**Corollary 1.** Let \( g \) and \( g^* \) be as defined in Theorems 1 and 2 respectively. Then for \( x > a \), \( g(x) = g^*(x) + C \), where \( C \) is a constant.

**Proof.** By Lemma 6 and Theorems 1 and 2.

**Corollary 2.** If \( f \) is a concave function satisfying the condition A, then the function \( g^* \) is a unique convex solution of (1) under the initial condition \( g(x_0) = y_0 \).

**Proof.** We only have to show that \( g^* \) is convex; the rest follows from Lemma 7. From (25) it follows easily that (with \( \lambda + \mu = 1 \))

\[
g^*(\lambda x + \mu y) - \lambda g^*(x) - \mu g^*(y) = - \sum_{k=0}^{+\infty} \left( f(\lambda x + k) + \mu(y + k) \right) - \lambda f(x + k) - \mu f(y + k),
\]

and therefore, concavity of \( f \) implies the convexity of \( g \). \( \Box \)

For \( |x - x_0| \leq 1 \), the series in Theorem 2 is of Leibniz type, so we may complement the expansion (25) by inequalities that follow immediately from Lemma 3, as in the next theorem.

**Theorem 2a.** Let \( f \) be a non-constant concave function that satisfies Condition A, and let \( g^* \) be defined as in Theorem 2. Then for any \( x > a \) and \( x_0 - 1 < x < x_0 \) the following inequalities hold:

\[
y_0 + (x - x_0) f(x_0) < g^*(x) < y_0 + (x - x_0) f(x_0) + f(x_0) - f(x)
\]

and, for all positive integers \( n \),

\[
g^*(x) > y_0 + (x - x_0) f(x_0) - \sum_{k=0}^{n} \left( f(x + k) - f(x_0 + k) \right) - (x - x_0) \left( f(x_0 + k + 1) - f(x_0 + k) \right)
\]

and
\[ g^*(x) < y_0 + (x - x_0)f(x_0) - \sum_{k=0}^{n} \left( f(x + k) - f(x_0 + k) \right) - (x - x_0)(f(x_0 + k + 1) - f(x_0 + k)) \]

\[-(f(x + n + 1) - f(x_0 + n + 1)).\]

For \(x_0 < x < x_0 + 1\), the above inequalities are reversed. \(\Box\)

The next theorem is proved in [10] as Theorem 3.7.

**Theorem 3.** Suppose that \(f\) satisfies the condition A and assume also that \(f\) is \(r\) times differentiable, with \(f^{(r)}(x)\) monotone for \(x\) large enough. Then the solution \(g\) of (1), introduced in Theorems 1 and 2 is also \(r\) times differentiable and we have

\[ g'(x) = \lim_{n \to +\infty} \left( f(x + n) - \sum_{k=0}^{n} f'(x + k) \right) \quad \text{and} \quad g^{(j)}(x) = -\sum_{k=0}^{+\infty} f^{(j)}(x + k) \quad (j \geq 2). \]

4. Applications to the Barnes’ Double Gamma function

For the purpose of this section, the Barnes \(G\)-function on \((0, +\infty)\) can be defined via Alexeiewsky theorem [2] as

\[ \log G(x) = -\frac{x - 1}{2} \log 2\pi - \frac{x(x - 1)}{2} + (x - 1) \log \Gamma(x) - \int_{1}^{x} \log \Gamma(t) \, dt. \]

It satisfies the functional equation

\[ \log G(x + 1) - \log G(x) = \log \Gamma(x), \quad G(1) = 1, \]

but it does not fit in the framework of Krull’s theory because \(\log \Gamma(x + 1) - \log \Gamma(x) = \log x\) does not converge to zero as \(x \to +\infty\). However, by differentiating in (26), we find that

\[ g(x) := (\log G(x))^\prime = (x - 1)(\Psi(x) - 1) + \frac{\log 2\pi - 1}{2}, \]

and

\[ g''(x) = 2\Psi'(x) + (x - 1)\Psi''(x) > 0 \quad \text{for} \quad x > 0, \]

where the positivity can be shown straightforward from the series representation of \(\Psi'\) and \(\Psi''\); also \(g''(x) \to 0\) monotonically as \(x \to +\infty\). So, the function \(g\) is a convex solution of the functional equation

\[ g(x + 1) - g(x) = \Psi(x), \quad g(1) = \frac{\log(2\pi) - 1}{2} \]

and the function \(\Psi\) is concave and satisfies condition A, whereas \(g\) satisfies condition B. Hence, we may apply results from the previous sections to the function \(g\) and integrate to obtain corresponding results for \(\log G\).

First note that by Corollary 2, the function \(g^\ast\) given by Theorem 2, is the unique solution of (29) that is either convex or \(g''(x) \to 0\) as \(x \to +\infty\) or only satisfies the condition B. Therefore,

\[ g(x) = (\log G(x))^\prime = C_0 - \gamma(x - 1) - \sum_{k=0}^{+\infty} \left( \Psi(x + k) - \Psi(1 + k) - \frac{x - 1}{k + 1} \right), \quad C_0 = g(1) = \frac{\log(2\pi) - 1}{2}. \]

This expansion is interesting in its own right; for example, using (30) and (28) we can find that

\[ \sum_{k=0}^{+\infty} \left( \Psi(x + k) - \Psi(1 + k) - \frac{x - 1}{k + 1} \right) = (x - 1)(\Psi(x) - 1 + \gamma). \]

Due to uniform convergence of series in (30), we may integrate on \([1, x]\) to obtain the following expansion for \(\log G\) in terms of logarithm and Digamma functions:
\begin{equation}
\log G(x) = C_0(x-1) - \gamma \frac{(x-1)^2}{2} - \sum_{k=0}^{\infty} \left( \log \Gamma(x+k) - \log \Gamma(1+k) - \Psi(1+k)(x-1) - \frac{(x-1)^2}{2(k+1)} \right),
\end{equation}
where we used the fact that $\log G(1) = 0$.

Now from (32) and the functional equation (27) we find that
\begin{equation}
\log \Gamma(x) = C_0 - \frac{\gamma(2x-1)}{2} - \sum_{k=0}^{\infty} \left( \log(x+k) - \Psi(1+k) - \frac{2x-1}{2(1+k)} \right)
\end{equation}
which gives a novel expansion for the Gamma function in terms of Digamma functions. With $x = 1$ it yields the determination of $C_0$ without assumption that it is equal to $(\log G(x))'$ at $x = 1$:
\begin{equation}
C_0 = \frac{\gamma}{2} + \sum_{k=0}^{\infty} \left( \log(1+k) - \Psi(1+k) - \frac{1}{2(1+k)} \right).
\end{equation}

Therefore, we have the following characterization of the Barnes $G$-function on $x \in (0, +\infty)$:

\textbf{Theorem 4.} Suppose that a real valued function $G$ is defined and differentiable on $x > 0$ and satisfies the functional equation
\[ \log G(x+1) - \log G(x) = \log \Gamma(x), \quad G(1) = 1. \]
If one of the following conditions is satisfied: (i) the function $x \mapsto (\log G(x))'$ is convex; (ii) $(\log G(x))'$ satisfies condition B; (iii) $(\log G(x))'''$ exists and converges to 0 as $x \to +\infty$, then $G$ is the Barnes $G$-function. \hfill \Box

Let us remark that a similar characterization is given in [5], but with the additional assumption that $G$ is meromorphic. On the other hand, since the expansion (32) can be extended to complex domain, defining a meromorphic function $G$, then we can prove that every function $G$ defined in complex domain, which satisfies conditions of Theorem 4 on $(0, +\infty)$, has to coincide with Barnes $G$-function for any complex argument.

By Corollary 1 we conclude that the expansion of $g = (\log G)'$ via Theorem 1 reads
\[ g(x) = C + \log \Gamma(x) - \frac{1}{2} \Psi(x) + \sum_{k=0}^{\infty} \left( \log(x+k) - \frac{1}{2}(\Psi(x+k+1) + \Psi(x+k)) \right), \]
where $C = 0$ by (34).

Double inequality (21) with $x_0 = 1$, $f(x) = \Psi(x)$ and $g(x) = (\log G(x))'$ becomes
\begin{equation}
\log \Gamma(t) - \frac{1}{2} \Psi(t) < (\log G(t))' < \log \Gamma(t) - \Psi(t) + \frac{1}{2} \log t.
\end{equation}
The absolute (and relative) errors in both inequalities monotonically converge to zero as $x \to +\infty$. As an illustration we note that for $t = 3$, with $g(3) = 0.2645$, the absolute errors in the left and right inequality in (35) are 0.03275 and 0.05516, and for $t = 100$, with $g(100) = 356.8$, the errors are 0.0008375 and 0.001666.

For $x > 1$, by integration in (35) with respect to $t \in [0, 1]$ and expressing $\int_1^x \log \Gamma(t) \, dt$ via (26), we get
\begin{equation}
\begin{align*}
(\log(2\pi) - 1)(x-1) - (x-1)^2 + (2x-3) \log G(x) \\
< 4 \log G(x) \\
< (\log(2\pi) - 1)(x-1) - (x-1)^2 + (2x-4) \log \Gamma(x) + x(\log x - 1) + 1.
\end{align*}
\end{equation}

Including more terms, via (22) and (23) and using the same method as above, we get, again for $x > 1$ and $n \geq 1$: 

\begin{equation}
4 \log G(x) > (\log(2\pi) - 2n - 1)(x - 2n - 1) - (x - 1)^2 + (2x - 3) \log \Gamma(x) \\
+ \sum_{k=0}^{n-1} ((2x + 2k - 1) \log(x + k) - (2k + 1) \log(k + 1) - 2 \log \Gamma(x + k) + 2 \log(k!))
\end{equation}

and

\begin{equation}
4 \log G(x) < (\log(2\pi) - 2n - 2)(x - 1) - (x - 1)^2 + (2x - 3) \log \Gamma(x) \\
+ \sum_{k=0}^{n-1} ((2x + 2k - 1) \log(x + k) - (2k + 1) \log(k + 1) - 2 \log \Gamma(x + k) + 2 \log(k!)) \\
+ (x + n) \log(x + n) - (1 + n) \log(1 + n) - \log \Gamma(x + n) + \log(n!).
\end{equation}

For $x < 1$ the inequalities above are reversed. Let us also note that by means of Theorem 2a, the expansion (32) can be transformed into a sequence of double inequalities (different than those discussed above) that hold for $x \in (0,2)$.

Finally, by an application of Theorem 3 with $g(x) = (\log G(x))'$ and $f(x) = \Psi(x)$, we obtain the following relations involving Psi and Poligamma functions, that seem not to have been recorded in the literature:

\begin{equation}
\Psi(x) + (x - 1)\Psi'(x) - 1 = \lim_{n \to +\infty} \left( \Psi(x + n) - \sum_{k=0}^{n} \Psi'(x + k) \right),
\end{equation}

\begin{equation}
2\Psi'(x) + (x - 1)\Psi''(x) = - \sum_{k=0}^{+\infty} \Psi''(x + k)
\end{equation}

and, in general, for $n \geq 2$, after some rearranging,

\begin{equation}
\sum_{k=1}^{+\infty} \Psi^{(n)}(x + k) = -(n\Psi^{(n-1)}(x) + x\Psi^{(n)}(x)).
\end{equation}

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