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REMARKS ON OSTROWSKI'S AND HADAMARD'S INEQUALITY

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Starting from Taylor's formula with an integral form of remainder, we derive an inequality which, as special cases, yields Ostrowski's and Hadamard's inequality. We further present a two sided version of Ostrowski's inequality for monotone functions and apply the result in derivation of an inequality for tails of a probability distribution concentrated on an interval.

1. AN INEQUALITY FOR DIFFERENTIABLE FUNCTIONS

Let $C_b^{(n+1)}[a,b]$ be the set of real valued functions g defined on $[a,b]$, n times differentiable at every point and with $g^{(n+1)}(x)$ that exists everywhere on $[a,b]$ except possibly finitely many points, such that $g^{(n+1)}$ is Riemann integrable over $[a,b]$, hence bounded. For a $g \in C_b^{(n+1)}[a,b]$, apply Taylor's expansion of order n arround an arbitrary $x \in (a,b)$, at points a and b respectively, to get

$$\begin{aligned} g(a) &= g(x) + g'(x)(a-x) + \sum_{k=2}^{n} \frac{g^{(k)}(x)}{k!}(a-x)^k + \frac{1}{n!}\int_x^a (a-\tau)^n g^{(n+1)}(\tau)\,\mathrm{d}\tau \\ g(b) &= g(x) + g'(x)(b-x) + \sum_{k=2}^{n} \frac{g^{(k)}(x)}{k!}(b-x)^k + \frac{1}{n!}\int_x^b (b-\tau)^n g^{(n+1)}(\tau)\,\mathrm{d}\tau, \end{aligned}$$

where for $n=1$, the sums are empty. By subtracting and dividing by $b-a$, we get

$$\frac{g(b)-g(a)}{b-a} - g'(x) = \sum_{k=2}^{n} \frac{g^{(k)}(x)}{k!} \cdot \frac{(b-x)^k - (a-x)^k}{b-a} + R_n, \tag{1}$$

where, after some work one can show that

$$R_n = \frac{(b-a)^n}{n!} \int_0^1 K(t) g^{(n+1)}(a + t(b-a))\,\mathrm{d}t \tag{2}$$

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and

$$K(t) = (-1)^n t^n I_{[0,(x-a)/(b-a)]}(t) + (1-t)^n I_{((x-a)/(b-a),1]}(t). \tag{3}$$

Let $B_{n+1} = \sup_{x\in[a,b]} |g^{n+1}(x)|$. Then for an even n we have that $K(t) \geq 0$ and hence

$$\begin{aligned} |R_n| &\leq B_{n+1}\frac{(b-a)^n}{n!}\int_0^1 K(t)\,\mathrm{d}t \\ &= \frac{(x-a)^{n+1}+(b-x)^{n+1}}{(b-a)(n+1)!}B_{n+1}. \end{aligned} \tag{4}$$

Note that if $g^{(n+1)}$ is of a constant sign on $[a,b]$ and n is even, then also R_n is of that sign.

For an odd n, the estimate (4) also holds; moreover, if $g^{(n+1)}(x) > 0$ for all $x \in [a,b]$ then from (2) and (3) we get

$$-\frac{(x-a)^{n+1}}{(b-a)(n+1)!}B_{n+1} \leq R_n \leq \frac{(b-x)^{n+1}}{(b-a)(n+1)!}B_{n+1}, \tag{5}$$

and if $g^{(n+1)} < 0$ then

$$-\frac{(b-x)^{n+1}}{(b-a)(n+1)!}B_{n+1} \leq R_n \leq \frac{(x-a)^{n+1}}{(b-a)(n+1)!}B_{n+1}, \tag{6}$$

Therefore, we have the following theorem.

Theorem 1. *Let $g \in C_b^{(n+1)}[a,b]$, $n \geq 1$. Further, let $B_{n+1} = \sup_{x\in[a,b]} |g^{(n+1)}(x)|$. For an arbitrary $x \in [a,b]$, define*

$$R_n(a,b,x;g) = \frac{g(b)-g(a)}{b-a} - g'(x) - \sum_{k=2}^{n}\frac{g^{(k)}(x)}{k!}\cdot\frac{(b-x)^k-(a-x)^k}{b-a}, \tag{7}$$

where the sum is void for $n = 1$. Then the integral representation (2) holds and we have the following inequality

$$|R_n(a,b,x;g)| \leq \frac{(x-a)^{n+1}+(b-x)^{n+1}}{(b-a)(n+1)!}B_{n+1}. \tag{8}$$

If $g^{(2m+1)}(x) \geq 0$ for all $x \in [a,b]$, then $R_{2m}(a,b,x;g) \geq 0$. If $n = 2m-1$, $m = 1,2,\ldots$ and $g^{2m}(x) \geq 0$ for all $x \in [a,b]$, then

$$-\frac{(x-a)^{2m}}{(b-a)(2m)!}B_{2m} \leq R_{2m-1}(a,b,x;g) \leq \frac{(b-x)^{2m}}{(b-a)(2m)!}B_{2m}. \tag{9}$$

The technique which was used here was also applied in [**3**] to obtain expressions for error terms in JENSEN's and some related inequalities.

In the next section we observe some special cases of Theorem 1.

2. SPECIAL CASES: OSTROWSKI'S AND HADAMARD'S INEQUALITIES

Let $f \in C_b^{(n)}[a,b]$ and let

$$g(x) = \int_a^x f(t)\,\mathrm{d}t \tag{10}$$

Then the expression (7) reads

$$\begin{aligned} R_n(a,b,x;g) = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f(x) \\ -\sum_{k=2}^{n} \frac{f^{(k-1)}(x)}{k!}\cdot\frac{(b-x)^k-(a-x)^k}{b-a} \end{aligned} \tag{11}$$

Let $M_n =$ ~~$B_{n+1} =$~~ $\sup_{x\in[a,b]} |f^{(n)}(x)|$. Theorem 1 for $n = 1$ gives Ostrowski's inequality:

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f(x)\right| \le \frac{(x-a)^2+(b-x)^2}{2(b-a)}M_1. \tag{12}$$

The bound in (12) is usually written in the form [**2**]:

$$(b-a)\left(\frac{1}{4}+\frac{(x-(a+b)/2)^2}{(b-a)^2}\right)M_1.$$

If $f' \ge 0$ on $[a,b]$, then by Theorem 1 we have

$$-\frac{(x-a)^2}{2(b-a)}M_1 \le \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f(x) \le \frac{(b-x)^2}{2(b-a)}M_1 \tag{13}$$

and if $f' \le 0$ on $[a,b]$, then

$$-\frac{(b-x)^2}{2(b-a)}M_1 \le \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f(x) \le \frac{(x-a)^2}{2(b-a)}M_1. \tag{14}$$

Suppose now that f is a positive unimodal function on $[a,b]$, with a unique maximum $f(x_0) = M$, $x_0 \in [a,b]$. Then $f'(x) \ge 0$ for $x \in [a,x_0)$ and $f'(x) \le 0$ for $x \in (x_0,b]$. From (2) and (3) we conclude that

$$0 \ge \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - M \ge -\frac{(x_0-a)^2+(b-x_0)^2}{2(b-a)}M_1, \tag{15}$$

that is,

$$M(b-a) - \frac{(x_0-a)^2+(b-x_0)^2}{2}M_1 \le \int_a^b f(t)\,\mathrm{d}t \le M(b-a). \tag{16}$$

For $n=2$, Theorem 1 yields

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f(x) - f'(x)\left(\frac{a+b}{2}-x\right)\right| \leq \frac{(x-a)^3+(b-x)^3}{6(b-a)}M_2. \tag{17}$$

Now, for $x=(a+b)/2$ and assuming that f is a convex function, we get the following two sided HADAMARD's inequality

$$0 \leq \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f\left(\frac{a+b}{2}\right) \leq \frac{(b-a)^2}{24}M_2 \tag{18}$$

From (12) with $x=(a+b)/2$ and (18), we obtain the following estimate:

$$0 \leq \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - f\left(\frac{a+b}{2}\right) \leq \min\left(\frac{b-a}{4}M_1, \frac{(b-a)^2}{24}M_2\right) \tag{19}$$

3. AN INEQUALITY FOR THE TAIL PROBABILITY

Let X be a continuous random variable concentrated on a finite interval $[a,b]$, i.e., $P(X \notin [a,b]) = 0$, where $0 \leq a < b$. Let f be the density of X; we assume that f is continuous on $[a,b]$ except possibly at finitely many points; let $M = \sup_{x\in[a,b]} f(x)$. Then the function $G(x) = P(X > x)$ is in $C_b^{(1)}[a,b]$, with $G'(x) = -f(x) \leq 0$, and the inequality (14) applied to the function G yields

$$-\frac{(b-x)^2}{2(b-a)}M \leq \frac{1}{b-a}\int_a^b P(X>t)\,\mathrm{d}t - P(X>x) \leq \frac{(x-a)^2}{2(b-a)}M. \tag{20}$$

For a non-negative random variable X we have the following expression for the expectation:

$$\mathrm{E}\,(X) = \int_0^{+\infty} P(X>t)\,\mathrm{d}t,$$

hence,

$$\int_a^b P(X>t)\,\mathrm{d}t = \mathrm{E}\,(X) - a$$

and (20) becomes

$$\frac{\mathrm{E}\,(X) - a - (x-a)^2M/2}{(b-a)} \leq P(X>x) \leq \frac{\mathrm{E}\,(X) - a + (b-x)^2M/2}{(b-a)}. \tag{21}$$

A similar inequality has been recently obtained in [**1**]. We can always take $a=0$ in (21), to obtain a simple inequality

$$\frac{\mathrm{E}\,(X) - x^2M/2}{b} \leq P(X>x) \leq \frac{\mathrm{E}\,(X) + (b-x)^2M/2}{b}, \tag{22}$$

which therefore holds for any positive continuous random variable X with a piecewise continuous density, such that $P(X > b) = 0$. If X is concentrated on an interval $[-c, b]$, for some $c > 0$, then the random variable $X' = X + c$ is concentrated on $[0, b + c]$ and (22) can be applied to X'.

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