Bayesian Selection for Heston Models with Volatilities Determined by Fourier Series Method

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Introduction

In the work of Polson and Roberts [4] a method for comparison between two diffusion models based on a continuous version of Bayes Factors is presented. Their method depends on an instantaneous volatility estimate, and it is well behaved only for constant diffusion coefficient. For a variable deterministic or stochastic volatility, the classical estimate of instantaneous volatility

$$\hat{\sigma}^2(t_i) = \frac{(X_{t_i} - X_{t_{i-1}})^2}{t_i - t_{i-1}}$$

becomes unstable and it has to be replaced with some other, more robust estimate. In this work we implement a Fourier series method of Malliavin and Mancino [2] for estimating the volatility function of an Ito process (see also [3, 5]). In a simulated model, we want to investigate performances of Bayesian model selection with stochastic volatility estimated by Fourier series method. We are primarily interested in showing that the Malliavin-Mancino method can be used as a robust estimate of instantaneous volatility in this setup. Our diffusions are models for the time evolution of a price S_t of some financial instrument.

Nonparametric Estimation of the Diffusion Coefficient by Fourier Series

We assume that the price follows an *Ito process* on an abstract probability space $(\Omega, \mathcal{F}, P)$ with continuous (by parts) volatility function:

$$dS_t = \mu(\omega, t)dt + \sigma(\omega, t)dB_t, \quad \forall t \in [0, T], \ \omega \in \Omega.$$

For these processes we define the *volatility function* as $\Sigma(\omega, t) = \sigma^2(\omega, t)$.
We are assuming that we have discrete measurements of the process in the time points $\{0 = t_1, ..., t_n = T\}$.
Define the *Fourier coefficients* of dS by

$$a_0(dS) = \frac{1}{T}\int_0^T dS_t, \quad a_k(dS) = \frac{2}{T}\int_0^T \cos\left(\frac{2\pi kt}{T}\right)dS_t$$
$$b_k(dS) = \frac{2}{T}\int_0^T \sin\left(\frac{2\pi kt}{T}\right)dS_t$$

and, as usual, the Fourier coefficients of the volatility function

$$a_0(\Sigma) = \frac{1}{T}\int_0^T \Sigma(t)dt, \quad a_k(\Sigma) = \frac{2}{T}\int_0^T \cos\left(\frac{2\pi kt}{T}\right)\Sigma(t)dt$$
$$b_k(\Sigma) = \frac{2}{T}\int_0^T \sin\left(\frac{2\pi kt}{T}\right)\Sigma(t)dt.$$

Based on this definitions we have the following Theorem from [3].

Theorem 1: For a fixed integer $n_0 > 0$, the Fourier coefficients of the volatility function are given, for all $k > 0$, by

$$a_0(\Sigma) = \lim_{N\to\infty} \frac{T}{N+1-n_0}\sum_{i=n_0}^{N} \frac{1}{2}(a_i^2(dS) + b_i^2(dS)),$$

$$a_k(\Sigma) = \lim_{N\to\infty} \frac{T}{N+1-n_0}\sum_{i=n_0}^{N} (a_i(dS)a_{i+k}(dS) + b_i(dS)b_{i+k}(dS)),$$

$$b_k(\Sigma) = \lim_{N\to\infty} \frac{T}{N+1-n_0}\sum_{i=n_0}^{N} (a_i(dS)b_{i+k}(dS) + b_i(dS)a_{i+k}(dS)),$$

where the convergence above is in probability. □
Having observed a path $S_t, 0 \le t \le T$, the Fourier coefficients of dS can be calculated using integration by parts, thus avoiding numerical differentiation of S_t:

$$a_k(dS) = \frac{2(S_T - S_0)}{T} - \frac{2k}{T}\int_0^T \sin\left(\frac{2\pi kt}{T}\right)S_t dt,$$

with a similar formula for $b_k(dS)$.
From **Theorem 1** one can reconstruct $\Sigma(t)$, for all $t \in [0, T]$, using *Féjer Formula* with the kernel $\varphi(x) = \frac{\sin^2(x)}{x^2}$. In fact,

$$\Sigma(t) = \lim_{M\to\infty} \Sigma_M(t),$$

where,

$$\Sigma_M(t) := \sum_{k=0}^{M} \varphi(\delta k)\left\{a_k(\Sigma)\cos\left(\frac{2\pi kt}{T}\right) + b_k(\Sigma)\sin\left(\frac{2\pi kt}{T}\right)\right\} \quad (1)$$

and δ is a smoothing parameter. In our application we will use $\delta = 0.04$.

Model Choice using Bayes Factor

From now on we assume that our Ito process is of the form

$$dS_t = (\theta f(\omega, t) + g(\omega, t))dt + \sigma(\omega, t)dB_t,$$

where θ is a unknown parameter and $f(\omega, t)$ and $g(\omega, t)$ are known functions.
Define $S^T = \{S_t \ ; \ 0 \le t \le T\}$. Let $P_{S^T|\theta}$ be the law of $S^T|\theta$ and let P_M be the measure induced by the process $dS_t = \sigma(\omega, t)dB_t$. So, the likelihood function is given by

$$l(\theta; X^T) = \frac{dP_{S^T|\theta}}{dP_M} = \exp\left\{\int_0^T \frac{\theta f(\omega, t) + g(\omega, t)}{\sigma^2(\omega, t)}dS_t - \right.$$
$$\left. -\frac{1}{2}\int_0^T \left[\frac{\theta f(\omega, t) + g(\omega, t)}{\sigma(\omega, t)}\right]^2 dt\right\},$$

and the log-likelihood can be written as

$$L(\theta) = \log l(\theta) =$$
$$= \int_0^T \frac{\theta f(\omega, t) + g(\omega, t)}{\sigma^2(\omega, t)}dS_t - \frac{1}{2}\int_0^T \left[\frac{\theta f(\omega, t) + g(\omega, t)}{\sigma(\omega, t)}\right]^2 dt.$$

Under the Bayesian paradigm the model is completely specified after the specification of a prior distribution for θ. This prior will be denoted as $p(\theta)$. We can calculate the marginal beliefs for the process as

$$P_{X^T}(A) = \int P_{X^T|\theta}(A)p(\theta)d\theta, \quad \forall A \in \sigma(X^T).$$

In order to calculate these marginal beliefs given a model we state the following Theorem [4]:
Theorem 2: Consider the model $\mathcal{M}_i$ that evolve according to

$$dS_t = (\theta f_i(\omega, t) + g_i(\omega, t))dt + \sigma_i(\omega, t)dB_t$$

and the prior distribution of θ given the model $\mathcal{M}_i$ is $N(\mu_i, \tau_i^2)$ for known hyperparameters μ_i and τ_i^2. Then the marginal beliefs $P_{X^T}(X^T|\mathcal{M}_i)$ are given by

$$2\log P_{X^T}(X^T|\mathcal{M}_i) = \frac{\tau_i^2 B_i^2 + 2\mu_i B_i - \mu_i^2 A_i}{1 + \tau_i^2 A_i} + 2C_i - \log(1 + \tau_i^2 A_i)$$

where

$$A_i = \int_0^T \left\{\frac{f_i(\omega, t)}{\sigma_i(\omega, t)}\right\}^2 dt,$$

$$B_i = \int_0^T \frac{f_i(\omega, t)}{\sigma_i^2(\omega, t)}dS_t - \int_0^T \frac{f_i(\omega, t)g_i(\omega, t)}{\sigma_i^2(\omega, t)}dt$$

$$C_i = \int_0^T \frac{g_i(\omega, t)}{\sigma_i(\omega, t)}dS_t - \frac{1}{2}\int_0^T \left\{\frac{g_i(\omega, t)}{\sigma_i(\omega, t)}\right\}^2 dt.$$

Now we can define the *Bayes Factor* between models $\mathcal{M}_0$ and $\mathcal{M}_1$ as

$$BF_{0,1}(X^T) = \frac{P_{X^T}(X^T|\mathcal{M}_0)}{P_{X^T}(X^T|\mathcal{M}_1)}.$$

In our application we will estimate $\sigma_i(\omega, t)$ using Equation (1).

An Example - Heston Model

In this section we will consider a slightly modified version of the Heston model [1] for the log-price $X_t = \log S_t$:

$$dX_t = \left(\theta - \alpha(e^{X_t} - \overline{X}(1 + \theta t)) - \frac{v_t}{2}\right)dt + \sqrt{v_t}dB_t,$$
$$dv_t = \kappa(\bar{v} - v_t)dt + \xi\sqrt{v_t}dW_t$$

with $dB_t dW_t = \rho dt$. Here θ is a parameter of the deterministic growth of the price process, and α is a mean-reverting parameter.
We simulated this model using the following values for parameters:

X_0	v_0	$\overline{X}$	κ	$\bar{v}$	ξ	θ	α	ρ
0	0.04	1	1	0.15	0.5	0.02	0.5	-0.7

In order to avoid negative (simulated) volatility the parameters were chosen to satisfy *Feller's condition*: $2\kappa\bar{v} \ge \xi^2$.

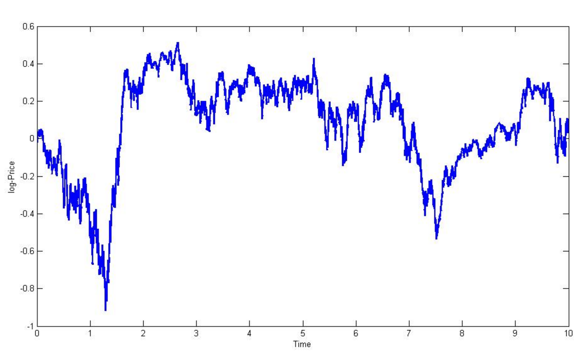

Figure 1: *Evolution of the log-price process*

To estimate the volatility we set, in **Theorem 1**, $N = 5000$ and $n_0 = 2$, and in equation (1), $M = 5000$.

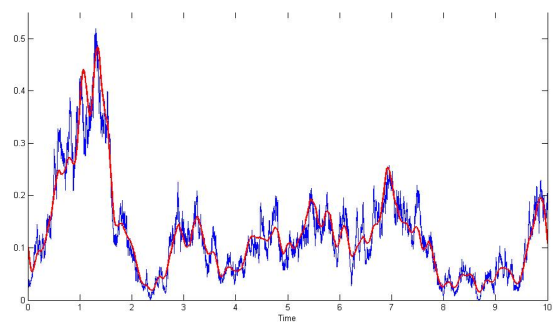

Figure 2: *True volatility function (blue), Estimated volatility function (red)*

Consider the models:

$$\mathcal{M}_0 \ : \ dX_t = \left(\theta - \frac{v_t}{2}\right)dt + \sqrt{v_t}dB_t, \quad \theta \sim N(0, 1)$$

$$\mathcal{M}_\alpha \ : \ dX_t = \left(\theta - \alpha(e^{X_t} - \overline{X}(1 + \theta t)) - \frac{v_t}{2}\right)dt + \sqrt{v_t}dB_t, \quad \theta \sim N(0, 1).$$

The Bayes Factor between $\mathcal{M}_\alpha$ and $\mathcal{M}_0$ is given by

$$BF_{\alpha,0}(X^T) = \frac{P_{X^T}(X^T|\mathcal{M}_\alpha)}{P_{X^T}(X^T|\mathcal{M}_0)}.$$

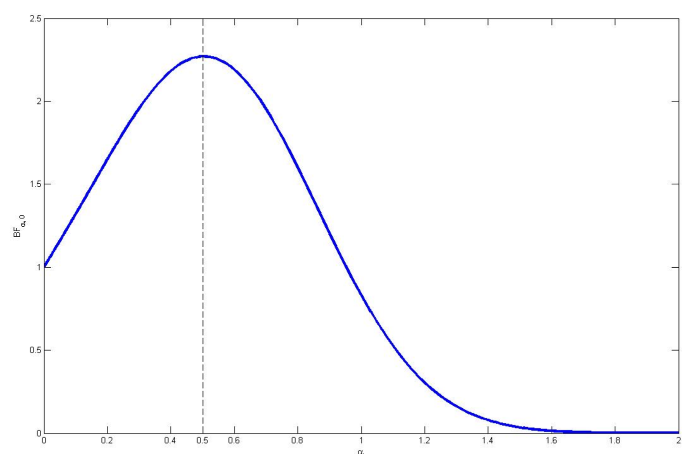

Figure 3: *Bayes Factor for different values of α with true $\theta = 0.02$.*

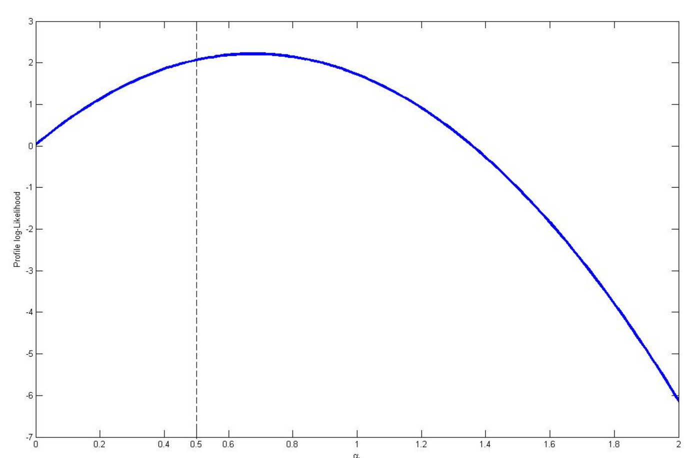

Figure 4: *Profile log-Likelihood function for $\theta = 0.02$ and different values of α*

Conclusion and Future Work

We presented a method developed by Malliavin and Mancino [2] for estimation of the volatility function of a semimartingale. We used this estimate to compare stochastic volatility models using generalized Bayes Factors [4]. The results appear to confirm the good properties of the described volatility estimator, for the case of stochastic volatility.

As a future work we attempt to use real data from financial market and to introduce a prior distribution to the parameter α. With real data it would be interesting to investigate the effect of the "market microstructure noise", as described in [6] in a context of realized volatility estimator.

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