Representations of error terms in Jensen's and some related inequalities with applications

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Error terms in inequalities

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Abstract. We give integral representations of error terms in the classical Jensen's inequality as well as in four related inequalities. Starting from these results, it is possible to derive various lower and upper asymptotic expansions for solutions of the functional equation $f(x+1) - f(x) = g(x)$. In particular, we present several results concerning the Gamma function.

1. Introduction

Let f be a real valued function defined on some interval I and let the derivative f' exists. Let F be an associated function in two variables, defined on I^2 by

$$F(x,y) = \frac{f(y) - f(x)}{y - x} \quad (x \neq y), \qquad F(x,x) = f'(x). \tag{1}$$

The following theorem is proved in [8]:

Theorem 0. If $x \mapsto f'(x)$ is continuous on I then the conditions $(A) - (E)$ below are equivalent:

(A) f' is convex on I.

(B) $f'\left(\dfrac{x+y}{2}\right) \leq F(x,y)$ for all $x, y \in I$,

(C) $F(x,y) \leq \dfrac{f'(x) + f'(y)}{2}$ for all $x, y \in I$,

(D) F is convex on I^2,

(E) $F(x+\varepsilon, y-\varepsilon) \leq F(x,y)$ for all $(x,y) \in I^2$, $x \leq y$ and $0 \leq \varepsilon \leq (y-x)/2$.

□

Integral versions of (B) and (C) are the well known Hadamard inequalities, conditions (A) and (D) can be expressed in terms of Jensen's inequalities for functions of one and two variables respectively, whereas (E) is equivalent to Schur-convexity of F [5]. In this paper we address the problem of estimating errors in these inequalities. We will show that the error in all five cases may be expressed via integral transforms of the third derivative with positive kernels. In fact, our starting point is a representation of the error in the classical Jensen's inequality (Theorem 1 in Section 2).

These results can be applied in several ways. First, a representation of the error term can serve as a basis for a comparison of inequalities that arise from convexity. Second, in some cases one can produce a sequence of inequalities that are asymptotically infinitely sharp, which leads to expansions with the knowledge of the sign of error at any step. This technique is particularly suitable for solutions f of the functional equation

$$f(x+1) - f(x) = g(x), \qquad x \geq a \tag{2}$$

(see [3] or [4] for a theory of this functional equation). In particular, equation (2) with $g(x) = \log x$ yields a unique convex solution $f(x) = \log \Gamma(x)$ and we will present several applications of our results to the theory of the Gamma function.

2. Representation of error terms

Theorem 1. Let f be a twice continuously differentiable function on an interval I. Then for all $\lambda \in [0,1]$ and all $(x,y) \in I^2$ we have

$$\begin{aligned} R_1(x,y,\lambda;f) &:= \lambda f(x) + (1-\lambda) f(y) - f(\lambda x + (1-\lambda)y) \\ &= (y-x)^2 \int_0^1 K_0(\lambda,t) f''((1-t)x + ty)\,\mathrm{d}t\ , \end{aligned} \tag{3}$$

where

$$K_0(\lambda,t) = \lambda t I_{[0,1-\lambda]}(t) + (1-\lambda)(1-t) I_{[1-\lambda,1]}(t), \tag{4}$$

I_A being the indicator of the set A.

Proof. Fix $x, y \in I$ and $\lambda \in [0,1]$ and let $c = \lambda x + (1-\lambda)y$. We may assume that $x < y$, without a loss of generality. By Taylor's formula with the remainder in the integral form, we have

$$f(x) = f(c) + f'(c)(x-c) + \int_c^x (x-\tau) f''(\tau)\,\mathrm{d}\tau\ , \tag{5}$$

$$f(y) = f(c) + f'(c)(y-c) + \int_c^y (y-\tau) f''(\tau)\,\mathrm{d}\tau\ . \tag{6}$$

Multiplying (5) by λ and (6) by $1-\lambda$ and adding, we get

$$\begin{aligned} \lambda f(x) + (1-\lambda) f(y) &= f(c) + \lambda \int_c^x (x-\tau) f''(\tau)\,\mathrm{d}\tau \\ &\quad + (1-\lambda) \int_c^y (y-\tau) f''(\tau)\,\mathrm{d}\tau \\ &= f(c) + \int_x^y G_0(\tau) f''(\tau)\,\mathrm{d}\tau, \end{aligned} \tag{7}$$

where

$$G_0(\tau) = \lambda(\tau - x)I_{[x,c]}(\tau) + (1-\lambda)(y-\tau)I_{[c,y]}(\tau).$$

Now the assertion follows from (8) upon letting $\tau = x + t(y-x)$, $0 \le t \le 1$.

Theorem 2. Let f be defined on I, with continuous third derivative f''' and let F be defined by (1). Then for all $(x,y) \in I^2$ we have

$$\begin{aligned} R_2(x,y;f) &:= F(x,y) - f'\left(\frac{x+y}{2}\right) \\ &= \frac{(y-x)^2}{2}\int_0^1 K_1(t)f'''((1-t)x+ty)\,\mathrm{d}t, \end{aligned} \tag{8}$$

where

$$K_1(t) = t^2 I_{[0,1/2]}(t) + (1-t)^2 I_{[1/2,1]}(t)\ . \tag{9}$$

Proof. For $(x,y) \in I^2$ being fixed, let $c = (x+y)/2$. Then by Taylor's formula we have

$$f(x) = f(c) + f'(c)\frac{x-y}{2} + f''(c)\frac{(x-y)^2}{8} + \frac{1}{2}\int_c^x (x-t)^2 f'''(\tau)\,\mathrm{d}\tau, \tag{10}$$

$$f(y) = f(c) + f'(c)\frac{y-x}{2} + f''(c)\frac{(y-x)^2}{8} + \frac{1}{2}\int_c^y (y-t)^2 f'''(\tau)\,\mathrm{d}\tau \tag{11}$$

From (10) and (11) it follows that

$$f(y) - f(x) = f'\left(\frac{x+y}{2}\right)(y-x) + \int_x^y G_1(\tau)f'''(\tau)\,\mathrm{d}\tau, \tag{12}$$

where

$$G_1(\tau) = \frac{(x-\tau)^2}{2}I_{[x,c]}(\tau) + \frac{(y-\tau)^2}{2}I_{[c,y]}(\tau).$$

Now (8) can be obtained by letting $\tau = x + t(y-x)$ in (12).

Theorem 3. Let f be defined on I, with continuous third derivative f''' and let F be defined by (1). Then for all $(x,y) \in I^2$ we have

$$\begin{aligned} R_3(x,y;f) &:= \frac{f'(x)+f'(y)}{2} - F(x,y) \\ &= \frac{(y-x)^2}{2}\int_0^1 t(1-t)f'''((1-t)x+ty))\,\mathrm{d}t\ . \end{aligned} \tag{13}$$

Proof. Start with the result of Theorem 1 applied to f' and integrate with respect to $\lambda \in [0,1]$ to get

$$\frac{f'(x)+f'(y)}{2} - F(x,y) = (y-x)^2\int_0^1 \left(\int_0^1 K_0(\lambda,t)\,\mathrm{d}\lambda\right) f'''(x+t(y-x))\,\mathrm{d}t.$$

Now it is straightforward to verify that

$$\int_0^1 K_0(\lambda, t)\,\mathrm{d}\lambda = \frac{t(1-t)}{2}$$

and (13) follows.

Theorem 4. Let F be defined by (1) on I^2, with a function f having a continuous third derivative f''' on I. Then for all $x_1, y_1, x_2, y_2 \in I$ and all $\lambda \in [0,1]$ the following holds:

$$\begin{aligned} R_4(x_1, y_1, x_2, y_2, \lambda; f) &:= \lambda F(x_1, y_1) + (1-\lambda)F(x_2, y_2) \\ &\qquad - F(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \\ &= \int_0^1 \Big((1-t)(x_2 - x_1) + t(y_2 - y_1)\Big)^2 \int_0^1 K_0(\lambda, u) \\ &\qquad \times f'''((1-t-u+tu)x_1 + (u-tu)x_2 + (t-tu)y_1 + tuy_2)\,\mathrm{d}u\,\mathrm{d}t\,. \end{aligned} \tag{14}$$

Proof. Let

$$\Delta_1 = y_1 - x_1, \quad \Delta_2 = y_2 - x_2, \quad \Delta_0 = \lambda(y_1 - x_1) + (1-\lambda)(y_2 - x_2),$$

$$x_0 = \lambda x_1 + (1-\lambda)x_2, \quad y_0 = \lambda y_1 + (1-\lambda)y_2\,.$$

Further, note that

$$\int_0^1 f'(x + t(y-x))\,\mathrm{d}t = \frac{1}{y-x}\int_x^y f'(\tau)\,\mathrm{d}\tau = \frac{f(y)-f(x)}{y-x} = F(x,y)$$

and so

$$\begin{aligned} &\lambda F(x_1, y_1) + (1-\lambda)F(x_2, y_2) - F(x_0, y_0) \\ &\qquad = \int_0^1 \big(\lambda f'(x_1 + t\Delta_1) + (1-\lambda)f'(x_2 + t\Delta_2) - f'(x_0 + t\Delta_0)\big)\,\mathrm{d}t \end{aligned} \tag{15}$$

An application of Theorem 1 to the integrand in (15) yields (14).

Theorem 5. Let F be defined by (1) on I^2, with a function f having a continuous third derivative f''' on I. Then for all $(x,y) \in I^2$, $x < y$ and all ε such that $0 \le \varepsilon \le (y-x)/2$, we have

$$\begin{aligned} R_5(x, y, \varepsilon; f) &= \\ F(x,y) - F(x+\varepsilon, y-\varepsilon) &= (y-x)^2 \int_0^1 (1-2t)^2 \int_0^1 K_0\left(1 - \frac{\varepsilon}{y-x}, u\right) \\ &\quad \times f'''\big((1-t-u+2ut)x + (t+u-2ut)y\big)\,\mathrm{d}u\,\mathrm{d}t\,. \end{aligned} \tag{16}$$

Proof. Introduce the notation $x_1 = x + \varepsilon$, $y_1 = y - \varepsilon$. Then

$$x_1 = \lambda x + (1-\lambda)y, \quad y_1 = (1-\lambda)x + \lambda y, \quad \text{where } \lambda = 1 - \frac{\varepsilon}{y-x}$$

and so

$$\begin{aligned} F(x_1, y_1) &= F(\lambda x + (1-\lambda)y, (1-\lambda)x + \lambda y) \\ &= F\big(\lambda(x,y) + (1-\lambda)(y,x)\big) \\ &= \lambda F(x,y) + (1-\lambda)F(y,x) - R = F(x,y) - R, \end{aligned}$$

where $R = R_4(x, y, y, x, 1 - \varepsilon/(y-x), f)$ is found in Theorem 4 and it is equal to the expression on the right hand side of (16). $\square$

Let $R_i, i = 1, \ldots, 5$ be as defined in Theorems 1–5. and let the function F be as defined by (1). The following two corollaries to Theorems 1-5 will serve as a basis for applications in Sections 3 and 4. Their proofs are straightforward and thus will be omitted.

Corollary 1. (i) Let f_1 and f_2 be twice continuously differentiable functions defined on an interval I, and suppose that $f_1''(x) \le f_2''(x)$ for all $x \in I$. Then for all $x, y \in I$ and $\lambda \in (0,1)$ we have that

$$R_1(x, y, \lambda; f_1) \le R_1(x, y, \lambda; f_2)$$

with strict inequality if $f_1''(t) < f_2''(t)$ for some t in the interval with endpoints x, y.

(ii) Let f_1 and f_2 be functions defined on I with continuous third derivatives, and suppose that $f_1'''(x) \le f_2'''(x)$ for all $x \in I$. Then

$$\begin{aligned} R_2(x, y; f_1) &\le R_2(x, y; f_2) \\ R_3(x, y; f_1) &\le R_3(x, y; f_3) \\ R_4(x_1, y_1, x_2, y_2, \lambda; f_1) &\le R_4(x_1, y_1, x_2, y_2, \lambda; f_2) \\ R_5(x, y, \varepsilon; f_1) &\le R_5(x, y, \varepsilon; f_2) \end{aligned}$$

for all $x, y, x_1, y_1, x_2, y_2 \in I$, $\lambda \in (0,1)$ and $\varepsilon > 0$ as specified in Theorem 5.

Remark. In fact, Corollary 1 (except the part concerning strict inequality) is an obvious consequence of linearity of R_i with respect to f, and can be derived directly from Theorem 0 under weaker assumptions. However, in our applications the conditions stated in Corollary 1 will always be satisfied.

Corollary 2. (i) Let $\{f_n\}$ be a sequence of twice continuously differentiable functions defined on an interval I, and suppose that $\lim_{n\to+\infty} f_n''(x) = 0$ for all $x \in I$. Then for all $x, y \in I$ and $\lambda \in (0,1)$ we have that

$$\lim_{n\to+\infty} R_1(x, y, \lambda; f_n) = 0.$$

(ii) Let $\{f_n\}$ be a sequence of functions defined on I with continuous third derivatives, and suppose that $\lim_{n\to+\infty} f_n'''(x) = 0$ for all $x \in I$. Then

$$\lim_{n\to+\infty} R_2(x,y;f_n) = 0, \qquad \lim_{n\to+\infty} R_3(x,y;f_n) = 0$$

$$\lim_{n\to+\infty} R_4(x_1,y_1,x_2,y_2,\lambda;f_n) = 0, \qquad \lim_{n\to+\infty} R_5(x,y,\varepsilon;f_n) = 0$$

for all $x, y, x_1, y_1, x_2, y_2 \in I$, $\lambda \in (0,1)$ and $\varepsilon > 0$ as specified in Theorem 5. $\square$

Let us also note that for $\lambda < 0$ and $x < y$ we have

$$y = \mu x + (1 - \mu z), \qquad \mu = \frac{1}{1-\lambda} \in (0,1), \quad z = \lambda x + (1-\lambda)y$$

and so

$$\lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y) = (\lambda - 1)R_1(x, z, \lambda/(\lambda-1); f).$$

In a similar way, one can see that for $\lambda > 1$ and $x < y$ we have

$$\lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y) = -\lambda R_1(z, y, 1/\lambda; f).$$

Therefore, we have

Corollary 3. Let $\{f_n\}$ be a sequence of twice continuously differentiable functions defined on an interval I and suppose that $\lim_{n\to+\infty} f''(x) = 0$ for all $x \in I$. Then

$$\lim_{n\to+\infty} (\lambda f_n(x) + (1-\lambda)f_n(y) - f_n(\lambda x + (1-\lambda)y)) = 0$$

for all $x, y \in I$ and $\lambda \in \mathbf{R}$ such that $\lambda x + (1-\lambda)y \in I$.

3. Applications to the functional equation (2)

In [3] the following result is proved: Let $x \mapsto g(x)$ be defined on $x \geq a$ and let $g(x) = g_1(x) + g_2(x)$, where g_1 is convex, g_2 is concave and

$$\lim_{x\to+\infty} (g_i(x+1) - g_i(x)) = 0, \qquad i = 1, 2.$$

Further, assume that g is r times differentiable ($r \geq 2$), with $x \mapsto f^{(r)}(x)$ monotone for x large enough. Then there exists a solution $x \mapsto f(x)$ of (2) such that

$$f^{(j)}(x) = -\sum_{k=0}^{+\infty} g^{(j)}(x+k) \qquad 1 \leq j \leq r, \tag{17}$$

and the series in (17) is uniformly convergent on $x \geq a$. Hence, $\lim_{x\to+\infty} g^{(j)}(x) = 0$ and by (17) the same holds for $f^{(j)}$. Further, if g is convex, then f is concave and conversely.

In this section we will typically observe solutions of (2) that are r times differentiable with $f^{(r)}(x) \geq 0$ or $f^{(r)}(x) \leq 0$ and $\lim_{x\to+\infty} f^{(r)}(x) = 0$, where $r = 2$ or $r = 3$. In the light of the above cited result, those requirements are not too restrictive.

If a_n and b_n are two sequences, let $a_n \lessapprox b_n$ stands for

$$a_n \leq b_n \quad \text{for all } n \quad \text{and} \quad \lim_{n\to+\infty}(a_n - b_n) = 0.$$

The notation $a_n \gtrapprox b_n$ is equivalent to $b_n \lessapprox a_n$.

In the rest of this section we understand the continuity of derivatives when it is necessary for applications of previous results.

3.1. Convex solutions: applications of Jensen's inequality. Let f be a convex solution of equation (2) on $x \geq a$ and suppose that $f''(x) \to 0$ as $x \to +\infty$.

Starting from

$$x + \beta = (1-\beta)x + \beta(x+1), \qquad x \geq a,\ \beta \in [0,1]$$

and applying Jensen's inequality, we get

$$\begin{aligned} f(x+\beta) &\leq (1-\beta)f(x) + \beta f(x+1) \\ &= (1-\beta)f(x) + \beta(g(x) + f(x)) \\ &= f(x) + \beta g(x) \end{aligned} \tag{18}$$

Replacing x with $x+n$, we get an inequality which, by Corollary 2, becomes infinitely sharp as $n \to +\infty$, i.e.,

$$f(x+n+\beta) - f(x+n) \lessapprox \beta g(x+n). \tag{19}$$

However, since f satisfies (2), then for $x \geq a$,

$$f(x+n) = f(x) + \sum_{k=0}^{n-1} g(x+k)$$

and (19) becomes

$$f(x+\beta) - f(x) \lessapprox \sum_{k=0}^{n-1} \big(g(x+k) - g(x+k+\beta)\big) + \beta g(x+n). \tag{20}$$

In the same way, starting from

$$x = \beta(x-1+\beta) + (1-\beta)(x+\beta), \qquad x \geq a+1-\beta,\ \beta \in [0,1]$$

we obtain

$$f(x+\beta) \geq f(x) + \beta g(x-1+\beta) \tag{21}$$

and further

$$f(x+\beta) - f(x) \gtrsim \sum_{k=0}^{n-1} \big(g(x+k) - g(x+k+\beta)\big) + \beta g(x+n-1+\beta). \tag{22}$$

The pair of expressions (20)-(22) give sharp bounds for the difference $f(x+\beta) - f(x)$ when $\beta \in [0,1]$ and for a large n. By Corollary 3, the asymptotics (with no knowledge of the sign of the error) holds regardless of the convexity of f and for all meaningful β. For instance, the following version of (20) holds for any twice continuously differentiable solution of (2) with $\lim_{x\to+\infty} f''(x) = 0$

$$f(x+\beta) - f(x) = \lim_{n\to+\infty} \left(\sum_{k=0}^{n-1} \big(g(x+k) - g(x+k+\beta)\big) + \beta g(x+n) \right),$$

for all $x \geq a$ and all real β such that $x+\beta \geq a$ and also $x \geq a$. Replacing x by x_0 and $x+\beta$ by x, we get the following expansion:

$$\begin{aligned} f(x) \quad &= \quad f(x_0) + \lim_{n\to+\infty} \left(\sum_{k=0}^{n-1} \big(g(x_0+k) - g(x+k)\big) + \right. \\ &\qquad \left. (x-x_0) g(x_0+n) \right), \end{aligned} \tag{23}$$

which is also derived in [3] by other means.

3.2. Solutions with a concave derivative: Hadamard's inequalities. Suppose that f is a solution of (2) with $f'''(x) \leq 0$ on $x \geq a$ and also suppose that $\lim_{n\to+\infty} f'''(x) = 0$. Then by Theorems 2 and 3 with $y = x+1$ and making use of (2) we have

$$f'(x) + \frac{1}{2} g'(x) \leq g(x) \leq f'(x+1/2), \qquad x \geq a,$$

and therefore

$$g(x-1/2) \leq f'(x) \leq g(x) - \frac{1}{2} g'(x), \qquad x \geq a+1/2.$$

This double inequality can be improved if we replace x by $x+n$ and use (2) in a similar way as in 3.1:

$$g(x+n-1/2) - \sum_{k=0}^{n-1} g'(x+k) \lessapprox f'(x)$$

$$\lessapprox \quad g(x+n) - \frac{1}{2}g'(x+n) - \sum_{k=0}^{n-1} g'(x+k)\ . \tag{24}$$

Note that the convergence in (24), as well as in other similar expressions is monotone if $f'''(x)$ monotonically converges to zero as $x \to +\infty$.

3.3. Convex solutions: Hadamard's inequalities. Let f be a convex solution of (2) on $x \geq a$ and suppose that $f''(x) \to 0$ as $x \to +\infty$. Now let

$$\varphi(x) = \int_{x_0}^{x} f(t)\,\mathrm{d}t, \qquad G(x) = \int_{x_0}^{x} g(t)\,\mathrm{d}t, \qquad x, x_0 \geq a.$$

Then integrating the relation

$$f(t+1) - f(t) = g(t)$$

with respect to $t \in [x_0, x]$, we get

$$\varphi(x+1) - \varphi(x) = G(x) + \varphi(x_0+1).$$

By assumption, $\varphi' = f$ is a convex function and by Theorems 2 and 3 we get

$$f(x+1/2) \leq G(x) + \varphi(x_0+1) \leq f(x) + \frac{1}{2}g(x), \quad \text{i.e.,}$$

$$G(x) - \frac{1}{2}g(x) + \varphi(x_0+1) \leq f(x) \leq G(x-1/2) + \varphi(x_0+1).$$

Further, an application of Corollary 2 yields

$$G(x+n) - \frac{1}{2}g(x) - \sum_{k=0}^{n-1} g(x+k) + C \lessapprox f(x)$$

$$\lessapprox \qquad G(x+n-1/2) - \sum_{k=0}^{n-1} g(x+k) + C, \tag{25}$$

where $C = \varphi(x_0+1)$. If x_0 can be chosen in such a way that $C = 0$, then from the left hand side of (25) we conclude that

$$f(x) = \lim_{n\to+\infty} \left(G(x+n) - \frac{1}{2}g(x) - \sum_{k=0}^{n-1} g(x+k) \right), \tag{26}$$

which also appears in [3]. However, (25) is much more informative than (26).

3.4. Solutions with a concave derivative: applications of Theorem 4 (concavity of function F). Suppose that f is a solution of (2) with $f'''(x) \le 0$ on $x \ge a$ and that $\lim_{n\to+\infty} f'''(x) = 0$. For a $\beta \in [0,1]$ and $x \ge a+1-\beta$ observe that

$$(x, x+1) = \beta(x-1+\beta, x+1) + (1-\beta)(x+\beta, x+1).$$

Then an application of Theorem 4 gives

$$F(x, x+1) \ge \beta F(x-1+\beta, x+1) + (1-\beta)F(x+\beta, x+1). \tag{27}$$

It is easy to see that

$$F(x,x+1) = g(x), \quad F(x-1+\beta, x+1) = \frac{f(x) - f(x+\beta) + g(x)}{2-\beta}, \quad \text{and}$$

$$F(x+\beta, x+1) = \frac{f(x) - f(x+\beta) + g(x)}{1-\beta}$$

and (27) yields

$$f(x+\beta) - f(x) \ge \frac{\beta}{2}(g(x) + g(x-1+\beta)). \tag{28}$$

Replacing x by $x+n$ and using Corollary 2, we get the expansion

$$\begin{aligned} f(x+\beta) - f(x) &\gtrsim \frac{\beta}{2}(g(x+n) + g(x+n-1+\beta)) \\ &+ \sum_{k=0}^{n-1}(g(x+k) - g(x+k+\beta)). \end{aligned} \tag{29}$$

3.5. Solutions with a concave derivative: applications of Theorem 5 (Schur-concavity of function F). Here we adopt the same assumptions on f as in 3.4. An application of Theorem 5 with $\varepsilon = 1$ gives

$$F(x-1, y+1) \le F(x,y), \qquad a+1 \le x < y,$$

or, after some work,

$$f(y) - f(x) \ge \frac{y-x}{2}(g(y) + g(x-1)), \qquad a \le x < y \tag{30}$$

By Corollary 2 we get the following expansion for the difference $f(y)-f(x)$:

$$f(y)-f(x) \quad \gtrsim \quad \frac{y-x}{2}(g(y+n)+g(x+n-1)) + \sum_{k=0}^{n-1}(g(x+k)-g(y+k)). \tag{31}$$

4. Applications to the Gamma function

Examples and applications presented in Section 4 can be specialized to the Gamma function. The function $x \mapsto \log\Gamma(x)$ is a (unique) convex solution of (2) with $g(x)=\log x$ and with a concave derivative $\Psi(x) = (\log\Gamma(x))'$. Many authors have considered inequalities for the ratio $\Gamma(x+\beta)/\Gamma(x)$ (Gautschi type inequalities, cf. [1]; for a more complete bibliography see [6]). A variety of methods have been used to invent those inequalities, although, as we showed in [6], most of them can be derived or improved by using only convexity results.

In asymptotic expansions related to the Gamma function, the following expression will frequently appear:

$$\Pi(x,\beta,n) = \frac{x(x+1)\cdots(x+n-1)}{(x+\beta)(x+\beta+1)\cdots(x+\beta+n-1)}.$$

From (18) with $f(x)=\Gamma(x)$ and $g(x)=\log x$ we get

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \le x^\beta, \qquad x>0, \beta\in[0,1].$$

This inequality was discovered by Wendel [9]. The corresponding expansion (20) reads

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \lesssim (x+n)^\beta \Pi(x,\beta,n). \tag{32}$$

Note that "$\sim$" in (32) is justified by the following reasoning: If $\lim_{n\to+\infty}(\log U_n(x) - \log U(x)) = 0$, where $U(x)$ does not depend on n, then

$$U_n(x)-U(x) = U(x)\left(e^{(\log U_n(x)-\log U(x))}-1\right) \to 0 \quad \text{as } n\to+\infty.$$

From (21) we obtain Gautschi's inequality [1]

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \ge (x-1+\beta)^\beta, \qquad x>0, \beta\in[0,1] \tag{33}$$

and the corresponding expansion (22) is

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \gtrapprox (x+n-1+\beta)^{\beta}\Pi(x,\beta,n).$$

In general, any inequality of the form

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \leq (\geq) B(x,\beta)$$

which is obtained using methods of previous sections, yields the expansion of the form

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \lessapprox (\ \gtrapprox\ ) B(x+n,\beta)\Pi(x,\beta,n).$$

Two more upper bounds can be derived from (28) and (30) respectively:

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \geq (x(x-1+\beta))^{\beta/2}, \qquad x>0, \beta\in[0,1] \tag{34}$$

and

$$\frac{\Gamma(y)}{\Gamma(x)} \geq ((x-1)y)^{(y-x)/2}, \qquad 0<x<y. \tag{35}$$

It is easy to see that (34) is sharper than (33) and (35); however, (35) is not restricted to $y-x\leq 1$.

From (24) we get expansions for the Digamma function:

$$\log(x+n-1/2) - \sum_{k=0}^{n-1}\frac{1}{x+n} \lessapprox \Psi(x) \lessapprox \log(x+n) - \frac{1}{2(x+n)} - \sum_{k=0}^{n-1}\frac{1}{x+n}\ . \tag{36}$$

Further, from (23) with $x_0 = 1$ we get an equivalent form of Euler's product:

$$\Gamma(x) \sim \frac{n!(n+1)^{x-1}}{x(x+1)\cdots(x+n-1)}.$$

Finally, (25) yields

$$K\cdot\frac{(x+n)^{x+n}e^{-(x+n)}}{x(x+1)\cdots(x+n-1)\sqrt{x+n}} \lessapprox \Gamma(x) \lessapprox K\cdot\frac{(x+n-1/2)^{x+n-1/2}e^{-(x+n-1/2)}}{x(x+1)\cdots(x+n-1)}\ , \tag{37}$$

where K is a constant. Letting $x=1$ in (37) and using Stirling's formula for the factorial, one gets $K=\sqrt{2\pi}$.

Bibliography

[1] W. Gautschi, *Some elementary inequalities relating to the gamma and incomplete gamma function*, J. Math. and Phys. **38** (1959), 77-81.

[2] J. Gurland, *An inequality satisfied by the Gamma function*, Skand. Aktuarietidiskr. **39** (1956), 171-172.

[3] W. Krull, *Bemerkungen zur Differenzengleichung* $g(x+1)-g(x) = \varphi(x)$, Math. Nachrichten **1** (1948), 365-376.

[4] M. Kuczma, "Functional equations in a single variable", Polish Scientific Publishers, Warszawa 1968.

[5] A. Marshall, I. Olkin, "Inequalities: Theory of Majorization and Its Applications", Academic Press, New York, 1979.

[6] M. Merkle, *Logarithmic convexity and inequalities for the Gamma function*, J. Math. Analysis Appl., **203** (1996), 369-380.

[7] M. Merkle, *Convexity, Schur-convexity and bounds for the Gamma function involving the Digamma function*, Rocky Mountain J. Math, to appear.

[8] M. Merkle, *Conditions for Convexity of a derivative and some applications to the Gamma function*, Aequ. Math., **55** (1998), 273-280.

[9] J.G. Wendel, *Note on the gamma function*, Am. Math. Monthly **55**(1948), 653-564.