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GAUSSIAN PROCESSES AND LINEAR INTERPOLATION*

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SUMMARY. We study Gaussian processes X_t such that the conditional expectation of X_t given values of X in a finite set S is a linear interpolation between two nearest values. For processes with independent increments, a characterization in terms of covariance function is given, together with a characterization of Wiener process. Using the theory of reciprocal processes, we give a characterization of Markov processes with linear interpolation property.

1. Introduction

Let $X_t, t \in T$, where T is a finite or infinite interval, be a centered Gaussian process, $EX_t = 0$ for all $t \in T$, and let $S = \{s_1, \ldots, s_n\}$ be a finite subset of T, $a \leq s_1 < s_2 < \ldots < s_n \leq b$. Let $E^S X_t = E(X_t | X_s, s \in S)$. In a Gaussian case, a conditioning is projection, therefore $E^S X_t$ is a linear combination of $X_s, s \in S$, where the coefficients depend on t:

$$E^S X_t = \sum_{i=1}^{n} C_i^S(t) X_{s_i}. \qquad \ldots (1)$$

We are interested in processes X_t with the property that $E^S X_t$ is the linear interpolation between $X_{s_{i-1}}$ and X_{s_i}, for $t \in [s_{i-1}, s_i]$, i.e.,

$$E^S X_t \doteq \frac{s_i - t}{s_i - s_{i-1}} X_{s_{i-1}} + \frac{t - s_{i-1}}{s_i - s_{i-1}} X_{s_i}. \qquad \ldots (2)$$

For example, a Wiener process W_t satisfies (2) on every interval $[a, b] \subset [0, +\infty)$ and for all finite S. This fact was used by Paul Lévy for a construction of Brownian motion (see Hida (1980), §2.3).

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Suppose that $a = s_1 < s_2 < \ldots < s_{n-1} < s_n = b$. For a Gaussian process X_t with a continuous covariance function that satisfies (2), we have

$$E\left(\int_a^b X_t, dt \,|X_s,\ s \in S\right) = \int_a^b E^S X_t dt$$

$$= \frac{s_2 - s_1}{2} X_{s_1} + \sum_{k=2}^{n-1} \frac{s_{k+1} - s_{k-1}}{2} X_{s_k} + \frac{s_n - s_{n-1}}{2} X_{s_n}. \qquad \ldots (3)$$

The right hand side of (3) is the trapezoidal rule approximation to the integral $\int_a^b X_t dt$. This means that the best estimate of $\int_a^b X_t dt$ based on observations of X_t on the set S is an approximation to the integral by the trapezoidal rule with knots in S. A review of possible applications of this and related properties can be found in Diaconis (1988).

In section 2 we characterize Gaussian processes that satisfy (2) for a fixed S, and in section 3 we consider Gaussian processes with independent increments satisfying (2) for every finite S. In this section we also give a characterization of a Wiener process. In section 4 we show that processes that satisfy (2) for all finite sets on an interval have to be so-called reciprocal processes, we give a characterization of Markov processes that satisfy (2) and we present several examples.

2. Characterization for a fixed set S

Multiplying (2) by $X_{s_k} (k = 1, 2, \ldots, n)$ and taking expectations, we obtain

$$EX_{s_k} X_t = \lambda_t EX_{s_k} X_{s_{i-1}} + (1 - \lambda_t) E_{s_k} X_{s_i}$$

and we conclude that

$$EX_{s_k} X_t = \alpha_k^i t + \beta_k^i, \quad t \in [s_{i-1}, s_i], \quad i = 2, \ldots, n, \qquad \ldots (4)$$

so, all functions $t \to EX_{s_k} X_t$ are piecewise linear and continuous.

Conversely, let (4) hold with some constants α_k^i, β_k^i and let $t \in [s_{i-1}, s_i]$. By (1) we have

$$E^S X_t = \sum_{j=1}^{n} A_j X_{s_j}. \qquad \ldots (5)$$

Let us multiply (5) by X_{s_k} for $k = 1, 2, \ldots, n$ and take expectations on both sides to obtain

$$EX_{s_k} X_t = \sum_{j=1}^{n} A_j EX_{s_k} X_{s_j}, \quad k = 1, 2, \ldots, n. \qquad \ldots (6)$$

If det $\|EX_{s_k}X_{s_j}\| \neq 0$, then the system (6) has a unique solution $\{A_j\}$. If (4) holds, then for every k we have

$$\begin{aligned} EX_{s_k}X_t &= \alpha_k^i t + \beta_k^i = \lambda_t(\alpha_k^i s_{i-1} + \beta_k^i) + (1-\lambda_t)(\alpha_k^i s_i + \beta_k^i) \\ &= \lambda_t EX_{s_k}X_{s_{i-1}} + (1-\lambda_t)EX_{s_k}X_{s_i} \end{aligned}$$

and this shows that a (unique) solution of (6) is given by

$$A_j = \begin{cases} \lambda_t & \text{if } j = k-1 \\ 1-\lambda_t & \text{if } j = k \\ 0 & \text{otherwise,} \end{cases}$$

which means that X_t satisfies (2). Therefore, we can formulate the following theorem.

Theorem 1. *Let $X_t, t \in T$ be a centered Gaussian process and let S be a given finite set of points in T, such that det $\|EX_{s_k}X_{s_j}\| \neq 0$. Then X_t satisfies (2) if and only if its covariance function satisfies (4).* □

In what follows, we will consider only Gaussian processes with non-singular covariance matrix, so we will assume that the determinant condition in Theorem 1 is always satisfied.

In order to produce some examples, we need an auxiliary result formulated in the following Lemma.

Lemma 1. *Let T be a set of reals and let $S = \{s_1, \ldots, s_n\}, n > 1$, be its fixed subset. Suppose that n real functions $C_1^S(\cdot), \ldots, C_n^S(\cdot)$ are defined on T and satisfy the condition*

$$C_i^S(s_j) = \delta_{ij}, \quad s_j \in S, \quad i, j = 1, \ldots, n. \qquad \ldots (7)$$

Then there exists a Gaussian process $X_t, t \in T$ such that (1) holds for given S and for all $t \in T$.

Proof. Let us define

$$\phi(s,t) = \sum_{i=1}^{n} C_i^S(s)C_i^S(t), \quad s, t \in T. \qquad \ldots (8)$$

This is a non-negative definite function, because for all $a_i \in C, i = 1, \ldots, m$ and $t_i \in T, i = 1, \ldots, m$, we have

$$\sum_{i=1}^{m}\sum_{j=1}^{m} a_i \overline{a}_j \phi(t_i, t_j) = \sum_{k=1}^{n} \left| \sum_{i=1}^{m} a_i C_k^S(t_i) \right|^2 \geq 0.$$

Therefore, there is a centered Gaussian process $X_t, t \in T$ with the covariance function ϕ. For such a process we have

$$E(X_t | X_{s_1}, \ldots, X_{s_n}) = \sum_{i=1}^{n} A_i(t) X_{s_i} \qquad \ldots (9)$$

To show that (1) holds, we will show that $A_i(t) = C_i^S(t)$. Indeed, multiplying both sides of (9) by X_{s_j} and taking expectations, we get

$$\phi(t, s_j) = \sum_{i=1}^{n} A_i(t)\phi(s_i, s_j).$$

By (7) we have that $\phi(s_i, s_j) = \delta_{ij}$ and therefore

$$\phi(t, s_j) = \sum_{i=1}^{n} A_i(t)\delta_{ij} = A_j(t).$$

On the other hand, by the definition of ϕ we have

$$\phi(t, s_j) = \sum_{k=1}^{n} C_k^S(t)C_k^S(s_j) = \sum_{k=1}^{n} C_k^S(t)\delta_{kj} = C_j^S(t),$$

which gives $A_i(t) = C_i^S(t)$ and the proof is completed. □

Examples. Let $S = \{u, v\}, u < v$. By Theorem 1 and the proof of Lemma 1, the function $(s,t) \to \phi(s,t)$ defined by

$$\phi(s,t) = \frac{v-t}{v-u}.\frac{v-s}{v-u} + \frac{t-u}{v-u}.\frac{s-u}{v-u} \quad s, t \in [u, v].$$

is the covariance of a centered Gaussian process that satisfies (2) for S as specified. It is easy to see that this process does not satisfy (2) for any other S.

In a similar way we can produce another interesting example. Let

$$\phi(s,t) = \left(\frac{v-t}{v-u} + \sin 2\pi\frac{v-t}{v-u}\right)\left(\frac{v-s}{v-u} + \sin 2\pi\frac{v-s}{v-u}\right) + \frac{t-u}{v-u}.\frac{s-u}{v-u}.$$

Then ϕ is the covariance function of a process which gives trapezoidal rule (3) on (u, v), but it does not satisfy (4) on (u, v).

3. Processes with independent increments

Let us assume that the increments of X_t are independent, $EX_t = 0$ for all t, and fix a finite set S and a time instant t such that

$$s_k < s_j < s_i < t < s_{i+1}.$$

For simplicity we will write X_k instead of X_{s_k} etc. By Theorem 1 and our assumptions, we have that

$$E(X_j - X_k)(X_t - X_i) = (a_j - a_k)(t - s_i) = 0. \qquad \ldots(10)$$

Since (10) holds for all $t \in [s_i, s_{i+1}]$, we have that $a_j = a_k$, and therefore,

$$EX_tX_{s_k} = \alpha t + b_k, \quad (s_k < s_i), \qquad \ldots(11)$$

where α and b_k are constants that may also depend on i.

Similarly, for $s_i < t < s_{i+1} < s_l < s_m$ we have

$$E(X_t - X_i)(X_m - X_l) = (a_m - a_l)(t - s_i) = 0, \qquad \ldots(12)$$

and from (12) we conclude that $a_m = a_l$ and

$$EX_tX_{s_m} = \beta t + b_m, \quad (s_m > s_{i+1}), \qquad \ldots(13)$$

where β and b_m are constants that may depend on i.

However, if (11) and (13) hold for every finite set S, then it is easy to see that the covariance must have the form

$$EX_tX_s = \begin{cases} \alpha t + b_1(s), & s \leq t \\ \beta t + b_2(s), & s \geq t \end{cases} \qquad \ldots(14)$$

where α, β are constants and b_1, b_2 some functions of s. By Theorem 1, the covariance function is continuous with respect to both arguments separately, which implies that both formulas in (14) should hold for $s = t$ and so $b_2(t) = b_1(t) + (\alpha - \beta)t$. Denoting b_1 by b we have

$$EX_tX_s = \begin{cases} \alpha t + b(s), & s \leq t \\ \beta t + (\alpha - \beta)s + b(s), & s \geq t. \end{cases} \qquad \ldots(15)$$

From (15), by $EX_tX_s = EX_sX_t$ we get $\alpha t + b(s) = \beta s + (\alpha - \beta)t + b(t)$, i.e., $b(t) = \beta t + b(s) - \beta s$. From this equality it follows that b is a linear function, $b(t) = \beta t + \gamma$. By substitution in (15) we have

$$EX_tX_s = \begin{cases} \alpha t + \beta s + \gamma & s \leq t \\ \alpha s + \beta s + \gamma & s \geq t. \end{cases}$$

The latter expression can be written as

$$EX_tX_s = \alpha(t \vee s) + \beta(t \wedge s) + \gamma. \qquad (16)$$

So, we have proved that (16) is implied by (2). It is easy to show that every process with a covariance as in (16) also satisfies (4), and therefore, by Theorem 1, (2) holds for such a process. As a result, we have the following

Theorem 2. *A Gaussian process X_t with independent increments and $EX_t = 0$ for all t, satisfies (2) for all finite sets S if and only if its covariance is of the form (16), for some real constants α, β, γ* □

To avoid repetition, we will give a name to the processes we are interested in.

Definition 1. We will say that a Gaussian process $X_t, t \in T$ is an LI process (linear interpolation process) on T if (2) holds for every finite set $S \subset T$.

Theorem 2 gives a characterization of centered *LI* processes with independent increments. For a non-centered case we have the following result.

Theorem 3. *A Gaussian process Y_t with independent increments is an LI process if and only if $Y_t = X_t + at + b$ where X_t is a centered Gaussian process with the covariance as in (16) and a and b are real constants.*

Proof. If X_t is a centered Gaussian process with the covariance of the form (16), it is an LI process by Theorem 2 and it can be easily seen that $Y_t = X_t + at + b$ is also an LI process.

Conversely, let Y_t be a Gaussian LI process with independent increments. By taking expectations on both sides of (2) with X replaced by $Y, S = \{u, v\}$ and $t \in (u, v)$, we get

$$\begin{aligned} EY_t &= \frac{EY_v - EY_u}{v - u} t + \frac{vEY_u - uEY_v}{v - u} \\ &= a(u, v)t + b(u, v). \end{aligned}$$

Since u, v are arbitrary, a and b are constants and $EY_t = at + b$. Then $X_t = Y_t - at - b$ is a centered Gaussian LI process. By Theorem 2, its covariance is of the form (16). □

Note that not all functions of the form (16) are covariances. For example, the function $(s, t) \to s \vee t$ is not a covariance (Cauchy - Schwartz inequality does not hold) on any $T \subset (0, +\infty)$.

Let us find the functions of the form

$$o(s, t) = \alpha(s \vee t) + \beta(s \wedge t) + \gamma \qquad \ldots (17)$$

that are covariances, i.e. positive definite functions on $T = [0, +\infty)$.

If we put $t_1 = t_2 = t$ in

$$\sum a_i a_j \phi(t_i, t_j) \geq 0$$

we get

$$(\alpha + \beta)t + \gamma \geq 0 \qquad \ldots (18)$$

Further, from Cauchy-Schwarz inequality we have

$$\alpha(\alpha t + \gamma) \leq \beta(\beta s + \gamma) \text{ for } s < t \qquad \ldots (19)$$

Suppose now that $T = [0, +\infty)$. Then for $t = 0$ in (18) we get $\gamma \geq 0$. Let $s = 0, t > 0$ in (19) to get

$$\alpha^2 t + \alpha\gamma \leq \beta\gamma$$

This may be possible only if $\alpha = 0$, and then by $\gamma \geq 0$ we have $\beta \geq 0$. Therefore, functions ϕ of the form (17) are covariances on $[0, +\infty)$ if any only if

$$\phi(s,t) = \beta(s \wedge t) + \gamma, \quad \beta, \gamma \geq 0. \qquad \ldots (20)$$

As a corollary to Theorem 3, we can give a characterization of a Wiener process.

Theorem 4. *A Gaussian process $Y_t, t \in [0, +\infty)$ with independent increments and $Y_0 = 0$ is an LI process if and only if*

$$Y_t = \sigma W_t + at,$$

where $\sigma > 0, a$ are real constants and W_t is the Wiener process.

Proof. By Theorem 3, a Gaussian process Y_t with independent increments is an LI process if and only if $Y_t = X_t + at + b$, where X_t is a Gaussian process with $EX_t = 0$ and the covariance function as in (20). By $Y_0 = 0$ we get $X_0 = -b$, and from $EX_0 = 0$ it follows $b = 0$ and $X_0 = 0$. Further, from (20) with $s = 0$ we get $0 = EX_tX_0 = \gamma$ for every $t > 0$, implying $\gamma = 0$. So, X_t is a Gaussian process with covariance $EX_tX_s = \beta(s \wedge t)$ and $X_0 = 0$, thus $X_t = \sigma W_t$, where $\sigma = \sqrt{\beta}$. □

4. Reciprocal processes and examples

The processes in the title have been an object of an intensive study by many authors and under several different names. It seems that they were firstly introduced by Bernstein in 1932, in connection with a problem posed by Scrödinger. Terms describing these and related processes include "Bernstein processes", "conditionally Markov processes" and "local Markov fields". These processes are also related to "G-maps" of Ylvisaker (1987) and" "(A,B) Markov processes" of Dynkin (1980). The term "reciprocal" is due to Jamison (1970).

Definition 2. The process $X_t, a < t < b$ is said to be reciprocal on (a, b) if

$$E(f(X_t)|X_{s_1}, \ldots, X_{s_n}) = E(f(X_t)|X_{s_i-1}, X_{s_i}), \qquad \ldots (21)$$

for every finite set $S = \{s_1, \ldots, s_n\} \subset (a, b), s_1 < \ldots < s_n$ and $t \in [s_{i-1}, s_i]$ and for any bounded Borel-measurable function f. □

As shown in the proof of Lemma 3 of Jamison (1970), in a Gaussian case, the requirement (21) of Definition 2 is equivalent to the following condition :

$$E(X_t|X_{s_1}, \ldots, x_{s_n}) = E(X_t|X_{s_{i-1}}, X_{s_i}) \qquad \ldots (22)$$

for $t \in [s_{i-1}, s_i]$.

Therefore, an LI process is necessarily a reciprocal process.

If a process is Markov one, then it is reciprocal; the converse is not true. In Timoszyk (1974) or Borisov (1982), the following characterization of Gaussian Markov processes can be found.

Lemma 2. *A function $R(s,t)$ defined on $T \times T$, where T is a finite or infinite interval on the real line, with $R(s,t) > 0$ everywhere on $T \times T$, is a covariance function of a Gaussian Markov process $X_t, t \in T$ with $EX_t = 0$ for $t \in T$ if and only if*

$$R(s,t) = G(s \wedge t)H(s \vee t), \qquad \ldots(23)$$

where the function $t \to G(t)/H(t)$ is a positive nondecreasing function on T. □

Starting from this result, we can characterize LI processes that are Markov:

Theorem 5. *A Gaussian process $X_t, t \in T = (0, +\infty)$ with $EX_t = 0$ for $t \in T$ and a positive covariance on $T \times T$ is an LI Markov process if and only if*

$$R(s,t) = EX_sX_t = (a(s \wedge t) + b)(c(s \vee t) + d), \quad (s,t \in T) \qquad \ldots(24)$$

where a, b, c, d are non-negative constants such that $ad - bc \geq 0$ and $a^2 + b^2 + c^2 + d^2 > 0$.

Proof. A Gaussian process with the covariance function as in (24) is a Markov process by Lemma 2 (condition $ad - bc \geq 0$ implies monotonicity of G/H). Further, R can be written in the form

$$R(s,t) = Ast + B(s \wedge t) + C(s \vee t) + D,$$

and it is not difficult to see that (4) holds for every finite set $S \in (0, +\infty)$. Thus, X_t is an LI process.

Conversely, let X_t be an LI Markov process with a positive covariance on $(0, +\infty)$. Since (4) holds for every S, the covariance function has to be of the form

$$R(s,t) = \alpha(s)t + \beta(s),$$

i.e.,

$$G(s \wedge t)H(s \vee t) = \alpha(s)t + \beta(s), \quad (s,t \in T). \qquad \ldots(25)$$

Let s_0 to be fixed and let $t \in (s_0, +\infty)$. Then by (25) we have

$$H(t) = \frac{\alpha(s_0)}{G(s_0)}t + \frac{\beta(s_0)}{G(s_0)}, \quad t \in (s_0, +\infty),$$

which is possible if and only if $H(t) = ct + d$ for some constants c, d. A similar argument shows that $G(s) = as + b$. Since G and H are continuous functions, and (by the assumption on covariance) do not have roots in $(0, +\infty)$, both functions have to be positive on $(0, +\infty)$, which is possible only if constants a, b, c, d are non-negative and $a^2 + b^2 + c^2 + d^2 > 0$. □

A characterization similar to the one of Theorem 5 can be worked out without the assumption that R is positive, relying on the corresponding characterization of Gaussian Markov processes in Timoszyk (1974) or Borisov (1982).

Examples. As an example for Theorem 5, a Gaussian process with $EX_sX_t = (s \wedge t)(s \vee t + 1)$ and $EX_t = 0, s, t > 0$ is an LI Markov process. This process does not have independent increments.

Brownian bridge on $(0,1)$ with the covariance $R(s,t) = (s \wedge t)(1 - s \vee t)$ is a Markov process but not an LI one.

In Carmichael *et al.* (1982) or (with an error) in Jamison (1970), a characterization of stationary reciprocal Gaussian processes in terms of covariance is given. From their results it follows that a unique stationary reciprocal process on $[0,b]$ with a linear covariance function is a process with

$$R(s,t) = EX_sX_t = 1 - \frac{1}{b}|s-t| = 1 - \frac{1}{b}(s \vee t - s \wedge t). \qquad \ldots (26)$$

This is a process with independent increments and it satisfies (16), so, by Theorem 2 it is an LI process. By Theorem 5 it is not a Markov process. Some properties of this process were investigated in Slepian (1961).

Here is an example from Mehr and McFadden (1965). The function

$$R(s,t) = 1 + 2(s \wedge t) - (s \vee t), \quad s,t \in [0,2]$$

is a covariance of a Gaussian process $X_t, t \in [0,2]$. Since it is of the form (4), X_t is an LI process, but not a Markov one.

References

Borisov, I.S. (1982). On a criterion for a Gaussian random process to be markovian, *Theor. Probab. Appl.* **27**, 863-865.

Carmichael J-P, Massé, J-C and Theodorescu R. (1982). Reciprocal Gaussian stationary processes on an interval, *C.R. Acad. Sc. Paris*, t. 295, Ser I, 291-293.

Cramér, H. and Leadbetter, M.R. (1967). *Stationary and Related Stochastic Processes*, Wiley.

Diaconis, P. (1988). Bayesian numerical analysis. In *Statistical Decision Theory and Related Topics IV*, S.S. Gupta, J.O. Berger eds, **1**, 163-175.

Dynkin, D.E. (1980). Markov processes and random fields, *Bull. Am. Math. Soc.* **3**, 975-999.

Hida, T. (1980). *Brownian Motion*, Springer Verlag.

Jamison, B. (1970). Reciprocal processes : The stationary gaussian case, *Ann. Math. Statist.*, **41**, 1624-1630.

—— (1974). Reciprocal processes, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **30**, 65-86.

—— (1975). The Markov processes of Schrödinger, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **32**, 323-331.

MEHR, C.B. and MCFADDEN, J.A. (1965). Certain properties of Gaussian processes and their first-passage times, *J. Roy. Statist. Soc.* **(VB) 27**, 505-522.

SLEPIAN, D. (1967). First passage time for a particular Gaussian processes, *Ann. Math. Statist.* **32**, 610-612.

TYMOSZYK, W. (1974). A characterization of Gaussian processes that are markovian, *Coll. Math.* **30**, f.1., 157-167.

YLVISAKER, D. (1987). Prediction and design, *Ann. Statist.* **15**, 1-19.

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