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Conditions for convexity of a derivative and some applications to the Gamma function

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Summary. We consider necessary and sufficient conditions for the convexity of a function $x \mapsto f'(x)$ in terms of some properties of the associated function of two variables $F(x,y) = (f(y)-f(x))/(y-x)$. In particular, we prove that f' is convex if and only if F is convex and if and only if F is Schur-convex. These results are applied to the theory of the Gamma function. We complement a characterization of the Gamma function due to H. Kairies and present some inequalities for the ratio of Gamma functions.

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1. Necessary and sufficient conditions for convexity

Let f be a differentiable function defined on an interval I. In this section we investigate a relationship between convexity (or concavity) of f' on I and convexity (or concavity) of the divided differences

$$F(x,y) = \frac{f(y)-f(x)}{y-x} \quad (x \neq y), \qquad F(x,x) = f'(x) \tag{1}$$

on I^2. We shall also consider Schur-convexity of F. For convenience, recall that a symmetric function $(x,y) \mapsto g(x,y)$ is Schur-convex on I^2 if and only if

$$g(x,y) \leq g(x-\varepsilon, y+\varepsilon)$$

for every $x, y \in I$, $\varepsilon > 0$ such that $x-\varepsilon, y+\varepsilon \in I$. A function g is Schur-concave if $-g$ is Schur-convex. For details on Schur-convexity see [11].

Let us consider the following statements:

(A) f' is convex on I.

(B) $f'\left(\dfrac{x+y}{2}\right) \leq F(x,y)$ for all $x, y \in I$,

(C) $\quad F(x,y) \le \dfrac{f'(x)+f'(y)}{2}$ for all $x,y \in I$,

(D) $\quad F$ is convex on I^2,

(E) $\quad F$ is Schur-convex on I^2,

and

(A′) $\quad f'$ is concave on I,

(B′) $\quad f'\left(\dfrac{x+y}{2}\right) \ge F(x,y)$ for all $x,y \in I$,

(C′) $\quad F(x,y) \ge \dfrac{f'(x)+f'(y)}{2}$ for all $x,y \in I$,

(D′) $\quad F$ is concave on I^2,

(E′) $\quad F$ is Schur-concave on I^2.

Our main result is the following.

Theorem 1. *If $x \mapsto f'(x)$ is continuous on I then the conditions* (A)–(E) *are equivalent and the conditions* (A′)–(E′) *are equivalent.*

For the proof of Theorem 1 we use the lemma below, which presents a slightly modified result from [11, 3.A].

Lemma 1. *Let $(x,y) \mapsto g(x,y)$ be a symmetric and continuous function on I^2. Suppose that both partial derivatives exist and are continuous on the set $\{(x,y) \in I^2 \mid x \ne y\}$. Then g is Schur-convex on I^2 if and only if*

$$\frac{\partial g(x,y)}{\partial y} - \frac{\partial g(x,y)}{\partial x} \ge 0 \quad \text{for } (x,y) \in I^2 \text{ such that } x < y. \tag{2}$$

Proof. Since g is continuous, it is Schur-convex if and only if for every fixed (x,y) in the interior of I^2, $x<y$, the function $\varepsilon \mapsto g(x-\varepsilon, y+\varepsilon)$ is increasing in $\varepsilon > 0$. After evaluation the derivative with respect to ε, one finds that the latter condition is equivalent to (2).

Proof of Theorem 1. (A)$\Longleftrightarrow$(B). By [15, p. 15], a continuous function g is convex on I if and only if

$$g(y) \le \frac{1}{2h}\int_{u-h}^{u+h} g(t)\ \mathrm{d}t$$

for all $u \in I$ and $h > 0$ such that $u \pm h \in I$. Letting here $u-h = x$, $u+h = y$ and $g = f'$, we get the desired assertion.

(A)$\Longleftrightarrow$(C). By [14, p. 39] or [15, p. 15], a continuous function g is convex on I if and only if

$$\frac{1}{y-x}\int_x^y g(t)\,\mathrm{d}t \le \frac{g(x)+g(y)}{2}$$

for all $x, y \in I$. Letting here $g = f'$, we get the assertion to be proved.

(A)$\Longleftrightarrow$(D). Trivially, by $f'(x) = F(x,x)$, we have that (D)$\Longrightarrow$(A). So, we need (A)$\Longrightarrow$(D) only. Now let f' be continuous and convex on I. Since F is a continuous function in two variables, to show its convexity it suffices to prove that [15, p. 215]

$$F(x,y) \leq \frac{1}{2}(F(x-\varepsilon, y-\eta) + F(x+\varepsilon, y+\eta)) \tag{3}$$

holds for any $x, y \in I$ and $\varepsilon, \eta > 0$ so that $x \pm \varepsilon, y \pm \eta \in I$. To this end, observe that

$$f(y) - f(x) = \int_x^y f'(t)\,\mathrm{d}t, \tag{4}$$

$$f(y-\eta) - f(x-\varepsilon) = \frac{y-x-\eta+\varepsilon}{y-x}\int_x^y f'(a_1 t + b_1)\,\mathrm{d}t, \tag{5}$$

$$f(y+\eta) - f(x+\varepsilon) = \frac{y-x+\eta-\varepsilon}{y-x}\int_x^y f'(a_2 t + b_2)\,\mathrm{d}t, \tag{6}$$

where

$$a_1 = \frac{y-x-\eta+\varepsilon}{y-x}, \quad a_2 = \frac{y-x+\eta-\varepsilon}{y-x}, \quad b_1 = \frac{x\eta - y\varepsilon}{y-x}, \quad b_2 = \frac{y\varepsilon - x\eta}{y-x}.$$

Now, since $(a_1 t + b_1 + a_2 t + b_2)/2 = t$ and f' is convex, we have

$$f'(t) \leq \frac{1}{2}(f'(a_1 t + b_1) + f'(a_2 t + b_2)).$$

By integration over $[x, y]$ and using (4)–(6), we obtain (3), as desired.

(E)$\Longleftrightarrow$(C). By Lemma 1, (E) is equivalent to

$$\frac{\partial F}{\partial x} \leq \frac{\partial F}{\partial y} \quad \text{for} \;\; x < y, \;\; x, y \in I$$

and it is easy to see that the latter condition is equivalent to (C).

The equivalence of conditions (A′)–(E′) follows from the above upon replacing f by $-f$. □

Theorem 1 can serve as a tool for producing many interesting inequalities. We shall consider some applications related to the Gamma function.

2. A characterization of Gamma function

The digamma function Ψ, defined by $\Psi(x) = (\log\Gamma(x))'$ can be expressed in terms of the series [1]

$$\Psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{+\infty} \frac{x}{n(n+x)} \qquad (x > 0),$$

where γ is the Euler constant. It follows that Ψ is concave on $x > 0$ and therefore, conditions (A′)–(E′) hold with $f = \log\Gamma$ on $I = (0, +\infty)$. In this and the next section we investigate some consequences of this fact.

In [7] it is proved that if g is concave on $(0, +\infty)$, $g(x+1) - g(x) = 1/x$ and $g(1) = -\gamma$ then $g(x) \equiv \Psi(x)$ for $x > 0$. In the light of Theorem 1, the requirement for concavity of $g = (\log\Gamma)'$ can be replaced by either of conditions (B′)–(E′) with $f = \log\Gamma$. Hence, in the next theorem we get a complement to the characterization in [7].

Theorem 2. *Suppose that f is a continuously differentiable real function defined on $(0, +\infty)$. If one of the conditions (A′)–(E′) is satisfied for f on $(0, +\infty)$ and*

$$f(x+1) - f(x) = \log x \qquad (x > 0),$$

then $f(x) = \log\Gamma(x) + C$, where C is an arbitrary real constant.

Proof. By Theorem 1, it suffices to assume that f' is concave. Then the statement can be proved using the mentioned result of [7], or by a result in [9]. We give an independent simple proof that relies on Bohr–Mollerup theorem [3].

Firstly, let us show that f' is a non-decreasing function. Indeed, suppose that $f'(a) > f'(b)$ for some $a < b$, $a, b > 0$. By the assumption, $f'(b+1) = f'(b) + 1/b > f'(b)$ and so we have that $f'(b) = \min\{f'(a), f'(b), f'(b+1)\}$, which is a contradiction to the concavity of f'. Therefore, f' is non-decreasing and consequently, f is a convex function. Let $h(x) = e^{f(x)-f(1)}$. Then h is positive and logarithmically convex on $(0, +\infty)$, $h(x+1) = xh(x)$ and $h(1) = 1$. By the Bohr–Mollerup theorem, $h(x) \equiv \Gamma(x)$ and the assertion follows. □

Note that omitting conditions (B′)–(E′) from the statement of Theorem 2 gives the result of Kairies [7].

3. Some inequalities for a ratio of the Gamma functions

Inequalities for the ratio $Q(x, \beta) = \Gamma(x+\beta)/\Gamma(x)$ with $x > 0$ and usually $\beta \in (0, 1)$, have been studied by many authors (see [2, 12] and references therein). Bounds for the ratio Q that involve the function Ψ and its derivatives have been investigated in [2, 4, 5, 6, 8, 13], using a variety of methods. In this section we present some new inequalities of this type, starting from (A$'$)–(E$'$) with $f = \log\Gamma$.

3.1. From (B$'$) and (C$'$) it follows that

$$\frac{1}{2}(\Psi(x)+\Psi(y)) \leq \frac{\log\Gamma(y)-\log\Gamma(x)}{y-x} \leq \Psi\left(\frac{x+y}{2}\right). \tag{7}$$

Letting $y = x+\beta$, $\beta > 0$, we get

$$\exp\left(\beta\frac{\Psi(x)+\Psi(x+\beta)}{2}\right) \leq Q(x,\beta) \leq \exp(\beta\Psi(x+\beta/2)). \tag{8}$$

The upper bound in (8) was also obtained in [8] by other means. In [13] we showed that the lower bound in (8) is closer than the lower bound in [8].

3.2. Since

$$(x, x+1+\beta) = (1-\beta)(x, x+1) + \beta(x, x+2),$$

(D$'$) yields

$$F(x, x+1+\beta) \geq (1-\beta)F(x, x+1) + \beta F(x, x+2) \qquad x > 0,\ \beta \in [0,1].$$

After an application of the recurrence relation $\Gamma(z+1) = z\Gamma(z)$ we get

$$Q(x,\beta) \geq \frac{x^{(1+\beta)(2-\beta)/2}(x+1)^{\beta(1+\beta)/2}}{x+\beta}. \tag{9}$$

Note the equality in (9) for $\beta = 0$ and $\beta = 1$. We shall show that for $\beta \in (0,1)$, the inequality (9) is sharper than Gautschi's inequality [5]

$$Q(x,\beta) \geq (x-1+\beta)^{\beta}. \tag{10}$$

Let us define a function φ by

$$\varphi(x) = \frac{(1+\beta)(2-\beta)}{2}\log x + \frac{\beta(1+\beta)}{2}\log(x+1) - \log(x+\beta) - \beta\log(x-1+\beta).$$

Then $\lim\limits_{x\to+\infty}\varphi(x) = 0$ and

$$\varphi'(x) = -\frac{\beta(1-\beta)(x^2+x+(2-\beta)(1+\beta))}{2x(x+1)(x-1+\beta)(x+\beta)} < 0 \quad \text{for} \quad x > 1-\beta \text{ and } \beta \in (0,1).$$

Therefore, we conclude that $\varphi(x) > 0$ for $x > 1-\beta$ and $\beta \in (0,1)$, which means that (9) is sharper than (10).

3.3. Further, from

$$(x, x+\beta) = (1-\beta)(x, x) + \beta(x, x+1)$$

and applying (D′) we obtain

$$Q(x, \beta) \geq x^{\beta^2} \exp(\beta(1-\beta)\Psi(x)), \qquad x > 0,\ \beta \in [0, 1]. \tag{11}$$

Using concavity of Ψ and inequality (14) below, it can be proved that this bound is closer than the lower bound in (8).

3.4. In a similar way, starting from

$$(x+\beta, x+\beta) = (1-\beta)(x+\beta, x) + \beta(x+\beta, x+1)$$

and applying (D′), we get

$$Q(x, \beta) \leq x^{-\beta^2/(1-2\beta)} \exp\left(\frac{\beta(1-\beta)}{1-2\beta}\Psi(x+\beta)\right), \quad x > 0,\ \beta < 1/2. \tag{12}$$

3.5. Condition (E′) implies

$$\frac{\log\Gamma(y) - \log\Gamma(x)}{y-x} \geq \frac{\log\Gamma(y+\varepsilon) - \log\Gamma(x-\varepsilon)}{y-x+2\varepsilon}$$

for $0 < x < y$ and $0 < \varepsilon < x$. In particular, replacing x by $x+\beta$ and letting $y = x+2\beta$ and $\varepsilon = \beta$, we obtain

$$\frac{\Gamma(x+3\beta)}{\Gamma(x)} \leq \left(\frac{\Gamma(x+2\beta)}{\Gamma(x+\beta)}\right)^2, \qquad x > 0,\ \beta > 0. \tag{13}$$

3.6. Let us now derive some bounds for the function Ψ. Letting $y = x+1$ in (7), we get

$$\Psi(x) + \frac{1}{2x} \leq \log x \leq \Psi\left(x + \frac{1}{2}\right),$$

whereof it follows

$$\log\left(x - \frac{1}{2}\right) \leq \Psi(x) \leq \log x - \frac{1}{2x}, \qquad x > 0. \tag{14}$$

4. An open question

The statements (A)–(E) presented in Section 1 can be reformulated as

(A) $x \mapsto F(x,x)$ is convex on I.

(B) $F\left(\frac{x+y}{2}, \frac{x+y}{2}\right) \le F(x,y)$ for all $x, y \in I$,

(C) $F(x,y) \le \frac{F(x,x)+F(y,y)}{2}$ for all $x, y \in I$,

(D) F is convex on I^2,

(E) F is Schur-convex on I^2.

where F is defined by (1). It would be interesting to investigate whether or not there exist symmetric functions F that can not be represented in the form (1) and such that all (or some) statements (A)–(E) are equivalent. Except (D)$\Longrightarrow$(A) and (D)$\Longrightarrow$(E), no other implication holds generally. Note that in the case considered in the present paper, $F(x,y)$ is completely determined by its values on the diagonal:

$$F(x,y) = \frac{1}{y-x}\int_x^y F(t,t)\,\mathrm{d}t.$$

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