INEQUALITIES FOR THE GAMMA FUNCTION VIA CONVEXITY

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Abstract. We review techniques based on convexity, logarithmic convexity and Schur-convexity, for producing inequalities and asymptotic expansions for ratios of Gamma functions. As an illustration, results for the Gautschi's and Gurland's ratio are presented, as well as asymptotic expansions for the Gamma function, along the lines of W. Krull's work. We argue that convexity-based techniques are advantageous over other methods, because they enable a comparision of inequalities, provide two transformations for their sharpening, and also yield two sided asymptotic expansions.

The Gamma function... is simple enough for juniors in college to meet, but deep enough to have called forth contributions from the finest mathematicians.

Philip Davis [7]

1 Introduction

This paper is devoted to the Euler's Gamma function of a real positive argument:

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} \mathrm{d}t, \qquad x > 0. \tag{1.1}$$

The importance of this function, which is classified as being *transcendentally transcendental* (Otto Hölder [13], Lee Rubel [33]) is enormous. It appears in almost every branch of Mathematics, and it is considered as an *almost elementary function.* Every mathematical or engineering software and even advanced calculators have a routine for calculation of $\Gamma(x)$.

The Gamma function first appeared in a letter of Leonard Euler to Goldbach in 1729, in the form of an infinite product:

$$x\Gamma(x) = \prod_{k=1}^{+\infty} \frac{k^{1-x}(k+1)^x}{x+k}. \tag{1.2}$$

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In this paper we give a survey of tools and techniques which can be used to produce inequalities for the Gamma function. We also present some examples of inequalities and asymptotic expansions.

By a famous theorem due to Bohr and Mollerup [3] and popularized by Emil Artin in his no less famous monograph [2] of 1931, Euler's gamma function is the unique solution to the functional equation $f(x+1) = xf(x)$, under the condition that $\log f$ is convex on $(0, +\infty)$. Hence, the logarithmic convexity is an essential property of the gamma function, and this property is shared by many of its generalizations in various directions. In spite of this well known fact, there is a wide spectrum of techniques and methods that have been used in the literature for producing inequalities, chiefly of the following types:

- Inequalities for the ratio $Q(x,\beta) = \dfrac{\Gamma(x+\beta)}{\Gamma(x)}$ (*Gautschi type*)
- Inequalities for the ratio $T(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma^2((x+y)/2)}$ (*Gurland type*)

The names are after W. Gautschi [9] and J. Gurland [10]. Gautschi type could as well be named after J. Wendel [38].

Most of the well known inequalities for the gamma and digamma functions can be derived and improved by means of logarithmic convexity, or related properties. Using convexity, we can also produce asymptotic expansions, expressed in terms of infinitely sharp inequalities. Our method is founded on certain general convexity results, as well as on integral representations of error terms in some classical and related inequalities, which will also be discussed.

Let us introduce two useful transformations; I call them a β-transform and a π_n-transform.

1.1 β-transform for Gautschi type

This transform has been known since Shanbhag [35]. The inequality

$$A(x,\beta) \le \frac{\Gamma(x+\beta)}{\Gamma(x)} \tag{1.3}$$

implies, replacing x by $x+\beta$ and β by $1-\beta$,

$$A(x+\beta, 1-\beta) \le \frac{\Gamma(x+1)}{\Gamma(x+\beta)},$$

and therefore

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \le \frac{x}{A(x+\beta, 1-\beta)}, \tag{1.4}$$

so, only the lower bound (1.3) is enough, or vice versa. We say that the inequality (1.4) is derived from (1.3) by a β-transform.

Two successive applications of the β-transform do not return the original bound; instead, it is equivalent to replacing x with $x+1$ and applying the recurrence $\Gamma(x+1) = x\Gamma(x)$. This leads us to another useful transform.

1.2 π_n-transform

This transform works for inequalities of both Gautcshi and Gurland type. It was firstly applied by B.R. Rao [30].

Let, for $n \geq 1$

$$\Pi(x, \beta, n) = \frac{x(x+1)\cdots(x+n-1)}{(x+\beta)(x+\beta+1)\cdots(x+\beta+n-1)}.$$

Start from the inequality of Gautschi type

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \leq B(x, \beta), \tag{1.5}$$

write it for $x+n$ and β and then apply the recurrence relation for the Gamma function, to obtain

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \leq B(x+n, \beta)\Pi(x, \beta, n). \tag{1.6}$$

Similarly, for an inequality of Gurland type

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma^2((x+y)/2)} \leq B(x, y) \tag{1.7}$$

one obtains

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma^2((x+y)/2)} \leq B(x+n, y+n)\rho(x, y, n), \tag{1.8}$$

where

$$\rho(x, y, n) = \frac{(x+y)^2(x+y+2)^2\cdots(x+y+2n-2)^2}{2^{2n}x(x+1)\cdots(x+n-1)y(y+1)\cdots(y+n-1)}. \tag{1.9}$$

It turns out that a π-transform sharpens inequalities; the reason will be obvious later.

Note that a β-transform can be thought of as a "square root" of π_1 transform.

We proceed with a survey of basic tools: convexity, logarithmic convexity, Schur-convexity and complete monotonicity.

2 Convexity

We begin with a survey of basic facts about convex functions. Throughout the text, I will denote an arbitrary real interval and $\bar{I}$ its closure.

A function f is convex on an interval I if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{2.1}$$

for all $x, y \in I$ and all $\lambda \in [0, 1]$. If the inequality in (2.1) is strict for all $\lambda \in (0, 1)$, then we say that f is strictly convex on I.

This definition can be easily extended to functions of more than one variable, if I is understood to be an arbitrary convex set, i.e., a set that together with x and y contains all points $\lambda x + (1-\lambda)y$ for $\lambda \in (0,1)$.

Inequality (2.1) is known as Jensen's inequality.

There are several equivalent forms of (2.1). For example, a function f is convex on I if and only if

$$\frac{f(x)-f(x_1)}{x-x_1} \le \frac{f(x_2)-f(x)}{x_2-x} \tag{2.2}$$

for all $x_1 < x < x_2$ in I [28, proof of Theorem 2 in 1.4.4]. Further, f is convex on I if and only if

$$\frac{f(y_1)-f(x_1)}{y_1-x_1} \le \frac{f(y_2)-f(x_2)}{y_2-x_2} \tag{2.3}$$

whenever $x_1 < y_1 \le y_2$ and $x_1 \le x_2 < y_2$ [21, 16B.3.a]. This condition means that the ratio $(f(v)-f(u))/(v-u)$ is increasing if the interval $[u,v]$ is being stretched to the right. It is sufficient to consider only a particular case $y_1 - x_1 = y_2 - x_2$ [21, 16B.3.a], i.e., a function f is convex on I if and only if

$$f(x+h) - f(x) \le f(y+h) - f(y) \tag{2.4}$$

for all $x < y$ in I and $h \ge 0$, such that $x+h, y+h \in I$. Even a further special case where $y = x+h$ is enough [21, 16B.3.a]: A function f is convex on I if and only if

$$f(x+2h) - 2f(x+h) + f(x) \ge 0 \tag{2.5}$$

whenever $x, x+h \in I$ and $h > 0$.

If a function f is convex on $\bar{I}$, then it is continuous on $\bar{I}$ and there exist left and right derivatives in each point $x \in I$. Moreover, $f'_-(x) \le f'_+(x)$ for each $x \in I$ [28, Theorem 1. in 1.4.4].

If a function f is differentiable on I, then from (2.4) it can be easily deduced that f is convex on I if and only if its derivative is an increasing function on I. If f is twice differentiable, then it is convex on I if and only if $f''(x) > 0$ on I.

If a function f satisfies

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} \tag{2.6}$$

for all $x, y \in I$, we say that f is J-convex (or convex in the sense of Jensen) on I [28]. A J-convex function on $\bar{I}$ need not be continuous, but if it is, then it must be convex in the sense of definition (2.1) [28]. Therefore, a continuous function f is convex on I if and only if (2.6) holds.

A function f is said to be concave on I if $-f$ is convex on I. The above formulas remain valid for a concave function f upon replacing $\le$ with $\ge$.

Sometimes it is convenient to consider functions that may take values $\pm\infty$ at some points of I. Then we apply the usual conventions, in particular $0 \cdot \pm\infty = 0$. Each finite valued function f, convex on $I \subset \mathbb{R}$ can be extended to a function $\hat{f}$, convex on $\mathbb{R}$ by

$$\hat{f}(x) = f(x) \quad \text{if } x \in I, \qquad \hat{f}(x) = +\infty \quad \text{otherwise.}$$

By [29, p.39] or [31, p.15], a continuous function g is convex on I if and only if

$$\frac{1}{y-x}\int_x^y g(t)\mathrm{d}t \leq \frac{g(x)+g(y)}{2}.$$

For a further study of convex functions see [31, 32, 36].

3 Logarithmic Convexity

A function f is said to be logarithmically convex (or log-convex) on I if $f(x) \geq 0$ for $x \in I$ and the function $x \mapsto \log f(x)$ is convex on I. Here we apply the convention $\log 0 = -\infty$.

Using the necessary and sufficient conditions for convexity given in the Introduction, it is easy to write the corresponding conditions for log-convexity. In particular, from (2.1) and (2.4) it follows that a nonnegative function f is log-convex if and only if

$$f(\lambda x + (1-\lambda)y) \leq (f(x))^{\lambda}(f(y))^{1-\lambda}, \qquad x, y \in I, \quad \lambda \in [0,1], \tag{3.1}$$

and if and only if

$$f(x+h)f(y) \leq f(x)f(y+h), \qquad x < y,\ x, y, x+h, y+h \in I,\ h \geq 0. \tag{3.2}$$

A nonnegative continuous function f is log-convex on I if and only if

$$f\left(\frac{x+y}{2}\right) \leq \sqrt{f(x)f(y)} \tag{3.3}$$

for each $x, y \in I$.

A positive twice differentiable function f is log-convex on I if and only if

$$f''(x)f(x) - f'(x)^2 \geq 0 \tag{3.4}$$

for all $x \in I$.

A function f is said to be log-concave on I if $f(x) \geq 0$ for all $x \in I$ and if the function $x \mapsto \log f(x)$ is concave on I. Clearly, f is log-concave on I if and only if $1/f$ is log-convex on I.

If f is log-convex on I, it is convex on I. If f is nonnegative and concave on I, it is log-concave on I. These assertions follow upon noticing that $t \mapsto e^x$ is an increasing convex and $t \mapsto \log t$ is an increasing concave function. Therefore, log-convexity is a stronger property of plain convexity and log-concavity is a weaker property than concavity. To illustrate the point, note that the function $x \mapsto e^{-x^2}$ is strictly convex and log-concave on $(1/\sqrt{2}, +\infty)$.

Since a sum of convex (concave) functions is also convex (concave), it follows that a product of log-convex (log-concave) functions on I is also a log-convex (log-concave) function on I.

If f_1, f_2 are log-convex functions on I, then their sum $f_1 + f_2$ is also a log-convex function [2]. By induction, it can be proved that the class of log-convex functions is closed under finite sums; by an extension to integrals one concludes that the function

$$F(x) = \int_a^b g(x,t)\mathrm{d}t$$

is a log-convex function on $x \in I$ if for each fixed $t \in (a,b)$ the function $x \mapsto g(x,t)$ is log-convex on I. As a classical example, one can show that the Gamma function

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1}\mathrm{d}t$$

is log-convex on $(0, +\infty)$.

However, this property does not hold for log-concave functions. Indeed, the sum of two log-concave functions may not be log-concave. This lack of additivity often makes proofs of log-concavity much more involved.

Many frequently encountered probability distribution functions are log-concave on their domain. For a survey see [26].

4 Schur Convexity

Given two vectors of dimension n, $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$, we say that x is majorized by y if

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]} \quad \text{for } k = 1, 2, \dots, n-1 \quad \text{and} \quad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i,$$

where $(x_{[1]}, x_{[2]}, \dots, x_{[n]})$ is a decreasing rearrangement of coordinates of x. If x is majorized by y, we write $x \prec y$. For example,

$$(\bar{x}, \bar{x}, \dots, \bar{x}) \prec (x_1, x_2, \dots, x_n), \qquad \text{where} \quad \bar{x} = (x_1 + \cdots + x_n)/n.$$

The notation and terminology was introduced in [11]. The theory and applications of the concept of majorization is studied in [21].

A function f of n variables is said to be Schur-convex on $A \subset \mathbb{R}^n$ if

$$x \prec y \implies f(x) \le f(y) \qquad \text{for each } x, y \in A. \tag{4.1}$$

A function f is said to be Schur-concave on $A \subset \mathbb{R}^n$ if

$$x \prec y \implies f(x) \ge f(y) \qquad \text{for each } x, y \in A. \tag{4.2}$$

Note that the term "convex" apparently has not much in common with the usual notion of convexity. The name was introduced by Schur [34], as opposed to convexity in the sense of Jensen. There are, however, many results showing a connection between these two notions.

Suppose that y is a permutation of x. Then $x \prec y$ and also $y \prec x$ and if f is Schur-convex then $f(x) \leq f(y)$ and $f(y) \leq f(x)$, i.e. $f(x) = f(y)$. Therefore, only symmetric functions (invariant to permutations) may be Schur-convex or Schur-concave.

If $x \prec y$ implies $f(x) < f(y)$ whenever $x, y \in A$ and x is not a permutation of y, we say that f is a strictly Schur-convex function. Strict Schur-concavity is defined analogously.

In case $n = 2$, there is a simple interpretation of Schur-convexity. Let $u = (u_1, u_2)$, $v = (v_1, v_2)$ and suppose that $u_1 \geq u_2$ and $v_1 \geq v_2$. Then $u \prec v$ means, by the definition,

$$u_1 \leq v_1, \qquad u_1 + u_2 = v_1 + v_2,$$

which may be possible if and only if

$$v_1 = u_1 + \varepsilon, \quad v_2 = u_2 - \varepsilon \qquad \text{for some } \varepsilon \geq 0.$$

Therefore, a function $f(x, y)$ is Schur-convex on A if and only if

$$f(x, y) \leq f(x - \varepsilon, y + \varepsilon) \tag{4.3}$$

for every $\varepsilon \geq 0$ and $x \leq y$, where $x, y, x - \varepsilon, x + \varepsilon \in A$. In words, $f(x, y)$ increases when the interval $[x, y]$ expands by equal amounts at both ends.

In general, let $A \subset \mathbb{R}^n$ be a convex and symmetric set (that is, it contains every chord connecting its two points and if it contains a point x, then it contains every y whose coordinates are a permutation of coordinates of x), with a nonempty interior. Then [21, 3.A.4] a continuously differentiable function f of n variables is Schur-convex on A if and only if it is symmetric and

$$(x_i - x_j)\left(\frac{\partial f(x)}{\partial x_i} - \frac{\partial f(x)}{\partial x_j}\right) \geq 0 \qquad \text{for all } x \in A. \tag{4.4}$$

Most often $A = I^n$, where I is an interval.

If a function g is convex on a real interval I, then [21, 3.C.1] the function

$$f(x) = \sum_{i=1}^{n} g(x_i)$$

is Schur-convex on I^n. For example, the function

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2$$

is Schur-convex on $\mathbb{R}^n$. Moreover, the inequality

$$\sum_{i=1}^{n} g(x_i) \leq \sum_{i=1}^{n} g(y_i)$$

holds for all continuous convex functions $g : \mathbb{R} \mapsto \mathbb{R}$ if and only if $x \prec y$. This result was obtained in [34] and is related to the much cited works of Tomić [37] and Weyl [39]; see also [21].

Let g be a continuous nonnegative function defined on an interval $I \subset \mathbb{R}$. Then

$$f(x) = \prod_{i=1}^{n} g(x_i), \qquad x \in I^n$$

is Schur-convex (strictly Schur-convex, Schur-concave, strictly Schur-concave) on I^n if and only if g is log-convex (strictly log-convex, log-concave, strictly log-concave) on I. For $n = 2$ it can be easily seen by comparing (4.3) for $f(x, y) = g(x)g(y)$ and the condition (3.2) for the function g.

In Section 7 it will be shown that the function

$$F(x, y) = \frac{\log \Gamma(y) - \log \Gamma(x)}{y - x}, \quad F(x, x) = (\log \Gamma(x))'$$

is Schur-concave on $\mathbb{R}^2$.

5 Complete monotonicity

A function f on $(0, +\infty)$ is completely monotone if it has derivatives of all orders and

$$(-1)^k f^{(k)}(t) \geq 0, \qquad t \in (0, +\infty),\ k = 0, 1, 2, \ldots$$

In particular, this implies $f \geq 0, f' \leq 0, f'' \geq 0$ and hence by (3.4), each completely monotone function on $(0, +\infty)$ is convex.

The function f is completely monotone on $(0, +\infty)$ if and only if [42]

$$f(x) = \int_0^{+\infty} e^{-xt} d\mu(t),$$

where $\mu(t)$ is nondecreasing and the integral converges for $0 < x < +\infty$.

The following result is proved in [14]: If $t \mapsto f(t)$ is completely monotone on $I = (0, +\infty)$, if $g(I) \subset I$ and if $x \mapsto g'(x)$ is completely monotone on I, then $x \mapsto f(g(x))$ is completely monotone on I.

In [8], a connection between complete monotonicity and majorization in the sense of Section 2 is discovered: Let α and β be vectors of dimension r with integer coordinates, and let f be completely monotone on $(0, +\infty)$. If $\beta \prec \alpha$, then for all $x \geq 0$

$$(-1)^{\alpha_1} f^{(\alpha_1)}(x)(-1)^{\alpha_2} f^{(\alpha_2)}(x) \cdots (-1)^{\alpha_r} f^{(\alpha_r)}(x)$$
$$\geq (-1)^{\beta_1} f^{(\beta_1)}(x)(-1)^{\beta_2} f^{(\beta_2)}(x) \cdots (-1)^{\beta_r} f^{(\beta_r)}(x).$$

By letting here $\alpha = (0, 2)$ and $\beta = (1, 1)$, we discover that every completely monotone function on $(0, +\infty)$ is log-convex.

As an example related to the Gamma function, let us examine the Digamma function

$$\Psi(x) = (\log \Gamma(x))'.$$

The explicit expression for the second derivative of $\log \Gamma$ reads [1]:

$$\Psi'(x) = (\log \Gamma(x))'' = \sum_{k=0}^{+\infty} \frac{1}{(x+k)^2}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$$

and therefore

$$\Psi^{(n)}(x) = (-1)^{n+1} n! \sum_{k=0}^{+\infty} \frac{1}{(x+k)^{n+1}}. \tag{5.1}$$

From (5.1) it follows that the function ψ' is completely monotone on $(0, +\infty)$. Several monotonicity results for functions related to the Gamma function are given in [14].

6 Characterizations of the Gamma function and Krull's Theory

The best known characterization of the Gamma function in terms of a functional equation, is due to Bohr and Mollerup:

Theorem 6.1 (Bohr-Mollerup ([3, 1922])). *If a function G is defined on $(0, +\infty)$, and satisfies:*

(1) $G(x+1) = xG(x)$, for all $x > 0$;

(2) $G(x)$ is log-convex;

(3) $G(1) = 1$,

then $G(x) \equiv \Gamma(x)$ for $x > 0$.

A parallel characterization of the Digamma function $\Psi(x) = (\log \Gamma(x))'$ is given in the next theorem.

Theorem 6.2 (Kairies ([15])). *If g is defined on $(0, +\infty)$ and*

(1) $g(x+1) - g(x) = 1/x$;

(2) $g(x)$ is concave on $(0, +\infty)$;

(3) $g(1) = -\gamma$,

then $g(x) \equiv \Psi(x)$ for $x > 0$.

Krull ([18], see also [19]) investigates the functional equation

$$f(x+1) - f(x) = g(x), \qquad x \geq a, \tag{6.1}$$

where f is an unknown function and g is given, such that

$$g \text{ is either convex or concave, or:} \tag{6.2}$$

$$g \text{ is the sum of a convex and a concave function, for } x \geq a$$

and

$$\lim_{x \to +\infty} \big(g(x+1) - g(x)\big) = 0. \tag{6.3}$$

By refining and complementing Krull's work, Bohr-Mollerup's theorem can be obtained as a special case.

Theorem 6.3. *Suppose that the equation (6.1) has a solution f that satisfies condition*

$$\lim_{x \to +\infty} \left(\frac{f(x+h_2) - f(x)}{h_2} - \frac{f(x) - f(x-h_1)}{h_1} \right) = 0, \quad h_1, h_2 < \delta \tag{6.4}$$

(without any other assumptions). Then all solutions that satisfy the condition (6.4) are of the form $f + C$, where C is an arbitrary constant.

Proof. Suppose that f_1 is another solution of (6.1) that satisfies condition (6.4) and let $\rho(x) = f_1(x) - f(x)$. Then ρ also satisfies the condition (6.4) and $\rho(x+1) = \rho(x)$ for $x \geq a$. Further, we have that

$$\begin{aligned} &\frac{\rho(x_0+h_2) - \rho(x_0)}{h_2} - \frac{\rho(x_0) - \rho(x_0-h_1)}{h_1} \\ &\qquad = \frac{\rho(x_0+n+h_2) - \rho(x_0+n)}{h_2} - \frac{\rho(x_0+n) - \rho(x_0+n-h_1)}{h_1} \to 0 \end{aligned}$$

as $n \to +\infty$, and we conclude that

$$\frac{\rho(x_0+h_2) - \rho(x_0)}{h_2} - \frac{\rho(x_0) - \rho(x_0-h_1)}{h_1} = 0$$

for any h_1, h_2 as specified above. Then by Section 2, we find that ρ is both convex and concave, and hence $\rho(x)$ is an affine function in a neighborhood of x_0; since x_0 is arbitrary, it follows that $\rho(x)$ is affine for $x > a$, and since ρ is periodic, it must be a constant. ∎

Corollary 6.4. *If a function f is defined on $(0, +\infty)$ and*

(1) $f(x+1) - f(x) = \log x$;

(2) $\lim_{x \to +\infty} f''(x) = 0$;

(3) $f(1) = 0$,

then $f(x) \equiv \log \Gamma(x)$ *for* $x > 0$.

Proof. It is easy to show that condition (6.4) is implied by the condition that $f''(x) \to 0$ as $x \to +\infty$. The rest follows from Theorem 6.3. ∎

7 Equivalent Conditions for Convexity of a Derivative

Theorem 7.1 (M. Merkle [24]). *Let* f *be defined on an interval* I*, with a continuous derivative* f'*. Define*

$$F(x,y) = \frac{f(y) - f(x)}{y - x} \quad (x \neq y), \qquad F(x,x) = f'(x). \tag{7.1}$$

Then the following are equivalent:

(A) f' *is convex on* I.

(B) $f'\left(\frac{x+y}{2}\right) \leq F(x,y)$ *for all* $x, y \in I$,

(C) $F(x,y) \leq \frac{f'(x) + f'(y)}{2}$ *for all* $x, y \in I$,

(D) F *is convex on* I^2,

(E) F *is Schur-convex on* I^2.

Also the following are equivalent:

(A') f' *is concave on* I.

(B') $f'\left(\frac{x+y}{2}\right) \geq F(x,y)$ *for all* $x, y \in I$,

(C') $F(x,y) \geq \frac{f'(x) + f'(y)}{2}$ *for all* $x, y \in I$,

(D') F *is concave on* I^2,

(E') F *is Schur-concave on* I^2.

From the above result, among other things, we may get another characterization of the Gamma function, as in the following theorem.

Theorem 7.2. *Suppose that* f *is a continuously differentiable real function defined on* $(0, +\infty)$*. If any of the conditions* $(A') - (E')$ *is satisfied with* f *on* $(0, +\infty)$ *and*

$$f(x+1) - f(x) = \log x \qquad (x > 0),$$

then $f(x) = \log \Gamma(x) + C$*, where* C *is an arbitrary real constant.*

8 Some Inequalities

From the expression (5.1) of Section 5, it follows that Ψ is concave for $x > 0$ and therefore, conditions $(A') - (E')$ of the previous section hold with $f = \log\Gamma$ on $I = (0, +\infty)$. In this and the next section we investigate some consequences of this fact.

8.1 From (B') and (C') it follows that

$$\frac{1}{2}(\Psi(x) + \Psi(y)) \le \frac{\log\Gamma(y) - \log\Gamma(x)}{y - x} \le \Psi\left(\frac{x+y}{2}\right). \tag{8.1}$$

Letting $y = x + \beta$, $\beta > 0$, we get

$$\exp\left(\beta\frac{\Psi(x) + \Psi(x+\beta)}{2}\right) \le Q(x,\beta) \le \exp(\beta\Psi(x + \beta/2)). \tag{8.2}$$

The upper bound in (8.2) was also obtained in [17] by other means. In [27] we showed that the lower bound in (8.2) is closer than a lower bound in [17].

8.2 Since

$$(x, x+1+\beta) = (1-\beta)(x, x+1) + \beta(x, x+2),$$

(D') yields

$$F(x, x+1+\beta) \ge (1-\beta)F(x, x+1) + \beta F(x, x+2) \qquad x > 0, \beta \in [0,1].$$

After an application of the recurrence relation $\Gamma(z+1) = z\Gamma(z)$ we get

$$Q(x,\beta) \ge \frac{x^{(1+\beta)(2-\beta)/2}(x+1)^{\beta(1+\beta)/2}}{x+\beta}. \tag{8.3}$$

Note the equality in (8.3) for $\beta = 0$ and $\beta = 1$.

8.3 From

$$(x, x+\beta) = (1-\beta)(x, x) + \beta(x, x+1)$$

and applying (D') we obtain

$$Q(x,\beta) \ge x^{\beta^2}\exp\left(\beta(1-\beta)\Psi(x)\right), \qquad x > 0, \beta \in [0,1]. \tag{8.4}$$

Using the concavity of Ψ and inequality (8.7) below, it can be proved that this bound is closer than the lower bound in (8.2).

8.4 In a similar way, starting from

$$(x+\beta, x+\beta) = (1-\beta)(x+\beta, x) + \beta(x+\beta, x+1)$$

and applying (D'), we get

$$Q(x,\beta) \le x^{-\beta^2/(1-2\beta)} \exp\left(\frac{\beta(1-\beta)}{1-2\beta}\Psi(x+\beta)\right), \qquad x>0, \beta<1/2. \tag{8.5}$$

8.5 The condition (E') implies

$$\frac{\log\Gamma(y)-\log\Gamma(x)}{y-x} \ge \frac{\log\Gamma(y+\varepsilon)-\log\Gamma(x-\varepsilon)}{y-x+2\varepsilon}$$

for $0<x<y$ and $0<\varepsilon<x$. In particular, replacing x by $x+\beta$ and letting $y=x+2\beta$ and $\varepsilon=\beta$, we obtain

$$\frac{\Gamma(x+3\beta)}{\Gamma(x)} \le \left(\frac{\Gamma(x+2\beta)}{\Gamma(x+\beta)}\right)^2, \qquad x>0, \beta>0. \tag{8.6}$$

8.6 Let us now derive some bounds for the function Ψ. Letting $y=x+1$ in (8.1), we get

$$\Psi(x)+\frac{1}{2x} \le \log x \le \Psi\left(x+\frac{1}{2}\right),$$

where from it follows

$$\log\left(x-\frac{1}{2}\right) \le \Psi(x) \le \log x - \frac{1}{2x}, \qquad x>0. \tag{8.7}$$

9 "Error terms" in Inequalities

In [23], we obtained the following integral representation related to Jensen's inequality:

Theorem 9.1. *Let f be a twice continuously differentiable function on an interval I. Then for all $\lambda\in[0,1]$ and all $(x,y)\in I^2$ we have*

$$\lambda f(x)+(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y) = (y-x)^2\int_0^1 K_0(\lambda,t)f''((1-t)x+ty)\mathrm{d}t\ ,$$

where

$$K_0(\lambda,t)=\lambda t I_{[0,1-\lambda]}(t)+(1-\lambda)(1-t)I_{[1-\lambda,1]}(t),$$

Since the kernel K_0 is positive, this representation yields three important tools, that are expressed in following corollaries.

Corollary 9.2. *If $0<f_1''<f_2''$ on (x,y) then Jensen's inequality obtained with f_1 is sharper than the one with f_2.*

Corollary 9.3. *If $0 < f''(x) \to 0$ as $x \to +\infty$, then Jensen's inequality becomes infinitely sharp as $x \to +\infty$.*

Corollary 9.4. *Let $\{f_n\}$ be a sequence of twice continuously differentiable functions defined on an interval I and suppose that $\lim_{n\to+\infty} f_n''(x) = 0$ for all $x \in I$. Then*

$$\lim_{n\to+\infty} (\lambda f_n(x) + (1-\lambda)f_n(y) - f_n(\lambda x + (1-\lambda)y)) = 0$$

for all $x, y \in I$ and $\lambda \in \mathbb{R}$ such that $\lambda x + (1-\lambda)y \in I$.

Corollary 9.2 enables comparison of inequalities that are derived from convexity, whereas Corollary 9.3 gives a way of producing asymptotically infinitely sharp inequalites, as we will demonstrate in the next section. Corollary 9.4 also provides asymptotic expansions, but without additional information of the sign of the error. Results analogous to Theorem 9.1, for remainders of expressions in $(A) - (E)$, and $(A') - (E')$ of Section 7, can be found in [23], in terms of the magnitude of f'''.

10 Back to Krull's Theory

If a_n and b_n are two sequences, let $a_n \lesssim b_n$ stand for

$$a_n \le b_n \quad \text{for all } n \quad \text{and} \quad \lim_{n\to+\infty} (a_n - b_n) = 0.$$

The notation $a_n \gtrsim b_n$ is equivalent to $b_n \lesssim a_n$.

Let f be a convex solution of Krull's equation (6.1) on $x \ge a$ and suppose that $f''(x) \to 0$ as $x \to +\infty$.

Starting from

$$x + \beta = (1-\beta)x + \beta(x+1), \qquad x \ge a,\ \beta \in [0,1]$$

and applying Jensen's inequality, we get

$$\begin{aligned} f(x+\beta) &\le (1-\beta)f(x) + \beta f(x+1) \\ &= (1-\beta)f(x) + \beta(g(x) + f(x)) \\ &= f(x) + \beta g(x) \end{aligned} \tag{10.1}$$

Replacing x with $x + n$, we get an inequality which, by Corollary 9.3, becomes infinitely sharp as $n \to +\infty$, i.e.,

$$f(x+n+\beta) - f(x+n) \lesssim \beta g(x+n). \tag{10.2}$$

However, since f satisfies (6.1), then

$$f(x+n) = f(x) + \sum_{k=0}^{n-1} g(x+k)$$

and (10.2) becomes

$$f(x+\beta) - f(x) \lesssim \sum_{k=0}^{n-1} \big(g(x+k) - g(x+k+\beta)\big) + \beta g(x+n). \tag{10.3}$$

In the same way, starting from

$$x = \beta(x-1+\beta) + (1-\beta)(x+\beta), \qquad x \geq a+1-\beta,\ \beta \in [0,1]$$

we obtain

$$f(x+\beta) \geq f(x) + \beta g(x-1+\beta) \tag{10.4}$$

and further

$$f(x+\beta) - f(x) \gtrsim \sum_{k=0}^{n-1} \big(g(x+k) - g(x+k+\beta)\big) + \beta g(x+n-1+\beta). \tag{10.5}$$

The pair of expressions (10.3) – (10.5) give sharp bounds for the difference $f(x+\beta) - f(x)$ when $\beta \in [0,1]$ and for a large n. By Corollary 9.4, the asymptotics (with no knowledge of the sign of the error) holds regardless of convexity of f and for all meaningful β. For instance, the following version of (10.3) holds for any twice continuously differentiable solution of (6.1) with $\lim_{x\to+\infty} f''(x) = 0$

$$f(x+\beta) - f(x) = \lim_{n\to+\infty} \left(\sum_{k=0}^{n-1} \big(g(x+k) - g(x+k+\beta)\big) + \beta g(x+n) \right),$$

for all $x \geq a$ and all real β such that $x+\beta \geq a$ and also $x \geq a$. Replacing x by x_0 and $x+\beta$ by x, we get the following expansion:

$$f(x) = f(x_0) + \lim_{n\to+\infty} \left(\sum_{k=0}^{n-1} \big(g(x_0+k) - g(x+k)\big) + (x-x_0) g(x_0+n) \right),$$

which is also derived in [18] by other means.

As an example related to the Gamma function, let $f(x) = \log\Gamma(x)$. This is a convex solution of Krull's equation with $g(x) = \log x$, and with $f''(x)$ monotonically decreasing to zero.

Then (10.1) gives

$$Q(x,\beta) = \frac{\Gamma(x+\beta)}{\Gamma(x)} \leq x^{\beta}, \tag{10.6}$$

which is Wendel's inequality [38]. Its improvement, by means of (10.3), reads:

$$Q(x,\beta) \lesssim (x+n)^{\beta}\,\Pi(x,\beta,n),$$

which is, in fact, a π_n-transform of (10.6).

11 How to Choose the Initial Function

In this section, we give a supply of convex and concave functions, to produce inequalities that involve the Gamma function.

Theorem 11.1 (M. Merkle, [25]). *Let B_{2k} be Bernoulli numbers and let L and R be generic notations for the following sums:*

$$L(x) = L_N(x) = -\sum_{k=1}^{2N} \frac{B_{2k}}{2k(2k-1)x^{2k-1}} \ , \quad (N = 1, 2, \ldots) \ , \quad L_0(x) = 0 \ .$$

$$R(x) = R_N(x) = -\sum_{k=1}^{2N+1} \frac{B_{2k}}{2k(2k-1)x^{2k-1}} \ , \quad (N = 0, 1, 2, \ldots)$$

(i) The functions

$$F_1(x) = \log\Gamma(x),$$
$$F_2(x) = \log\Gamma(x) - x\log x,$$
$$F_3(x) = \log\Gamma(x) - \left(x - \frac{1}{2}\right)\log x,$$
$$F_4(x) = \log\Gamma(x) - \left(x - \frac{1}{2}\right)\log x - \frac{1}{12x} + \frac{1}{360x^3}$$

and

$$F(x) = \log\Gamma(x) - \left(x - \frac{1}{2}\right)\log x + L(x)$$

are convex on $x > 0$.
(ii) The functions

$$G_1(x) = \log\Gamma(x) - \left(x - \frac{1}{2}\right)\log x - \frac{1}{12x},$$

$$G_2(x) = \log\Gamma(x) - \left(x - \frac{1}{2}\right)\log x - \frac{1}{12x} + \frac{1}{360x^3} - \frac{1}{1260x^5}$$

and

$$G(x) = \log\Gamma(x) - \left(x - \frac{1}{2}\right)\log x + R(x)$$

are concave on $x > 0$.

Including more terms has the effect of decreasing the absolute value of the second derivative, and hence, yields a sharper inequality (but, more complicated). In the limit, functions F and G become $(\log 2\pi)/2 - x$, in accordance with a well known asymptotic expansion:

$$\log\Gamma(x) \sim \left(x - \frac{1}{2}\right)\log x - x + \frac{1}{2}\log 2\pi + \sum_{k=1}^{+\infty} \frac{B_{2k}}{2k(2k-1)x^{2k-1}}$$

12 Inequalities for Gautschi's Ratio

As a starting point, let us see what one can obtain from the plain log-convexity of the Gamma function. By Jensen's inequality with $\varphi(x) = \log \Gamma(x)$, we find

$$\varphi(x) \le \beta\varphi(x-1+\beta) + (1-\beta)\varphi(x+\beta) \qquad (0 \le \beta \le 1)\ . \tag{12.1}$$

That is,

$$\Gamma(x) \le \Gamma^{\beta}(x-1+\beta)\Gamma^{1-\beta}(x+\beta) = \frac{\Gamma^{\beta}(x+\beta)}{(x-1+\beta)^{\beta}}\Gamma^{1-\beta}(x+\beta)$$

and this gives the well known Gautschi inequality

$$Q(x,\beta) = \frac{\Gamma(x+\beta)}{\Gamma(x)} \ge (x-1+\beta)^{\beta}\ . \tag{12.2}$$

Eleven years before Gautschi, J.G.Wendel published a note [38] on the Gamma function, containing inequalities ($x \ge 1, \beta \in [0,1]$)

$$\frac{x}{(x+\beta)^{1-\beta}} \le Q(x,\beta) \tag{12.3}$$

and

$$Q(x,\beta) \le x^{\beta} \tag{12.4}$$

J.T. Chu in the article [6], published in 1962, gives the following result:

$$\sqrt{\frac{n-1}{2}}\sqrt{\frac{2n-3}{2n-2}} < \frac{\Gamma(\frac{n}{2})}{\Gamma\left(\frac{n}{2}-\frac{1}{2}\right)} < \sqrt{\frac{n-1}{2}}\sqrt{\frac{2n-2}{2n-1}}, \tag{12.5}$$

($n = 2, 3, \ldots$), which, after letting $x = \frac{n-1}{2}$, becomes

$$\sqrt{x-\frac{1}{4}} < \frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x)} < \frac{x}{\sqrt{x+\frac{1}{4}}}. \tag{12.6}$$

Note that the upper bound may be obtained from the lower one by the $\beta-$ transform. Chu indicates that, for $x = 1, 2, \ldots$, there is an improvement in the lower bound, which we will write as

$$\sqrt{x-\frac{1}{4}+\frac{1}{(4x+2)^2}} < \frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x)}. \tag{12.7}$$

In 1967, Boyd [5] gives inequalities in the same spirit. The lower bound in our notation reads

$$\sqrt{x-\frac{1}{4}+\frac{1}{32x+16}} < \frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x)}, \tag{12.8}$$

for $x = m + \frac{1}{2}$, $m = 1, 2, \ldots$, and an upper bound can be found from (12.8) and the $\beta-$ transform.

Finally, Lazarević and Lupaş' result [20] from 1979 reads:

$$\left(x - \frac{1-\beta}{2}\right)^{\beta} \leq \frac{\Gamma(x+\beta)}{\Gamma(x)}, \tag{12.9}$$

for $x > \frac{1-\beta}{2}$ and $\beta \in [0, 1]$. This inequality was rediscovered by Kershaw [17] in 1983.

Applying the $\beta-$ transform, we can find the corresponding upper bound:

$$\frac{\Gamma(x+\beta)}{\Gamma(x)} \leq \frac{x}{(x + \frac{\beta}{2})^{1-\beta}} \tag{12.10}$$

Lazarević-Lupaş' inequality (12.9) is, for $x > (\beta^2 + 3)/4$ (and therefore, for $x \geq 1$ and every $\beta \in (0, 1)$) sharper than Wendel's inequality (12.3), which is sharper than Gautschi's (12.2).

As we can see, (12.9) concides with (12.6) where the latter holds, but it is much more general than (12.6). On the other hand, (12.8) is sharper than (12.9) for a particular choice of x and β. An inequality which generalizes and sharpens all the mentioned inequalities is proved in [26]:

For any $x \geq (1-\beta)/2$ and $\beta \in [0, 1]$,

$$Q(x, \beta) \geq \left(x - \frac{1-\beta}{2} + \frac{1-\beta^2}{24x+12}\right)^{\beta}, \tag{12.11}$$

with equality if and only if $\beta = 0$ or $\beta = 1$. This inequality is derived as a simplified version of the inequality that is obtained from concavity of the function G_1 of the previous section.

13 Inequalities and Expansions for Gurland's Ratio

The first result about the ratio of Gamma functions

$$T(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma^2\left(\frac{x+y}{2}\right)}, \quad x, y > 0,$$

appeared in 1956, in John Gurland's paper [10], where the following inequality was presented:

$$\frac{\Gamma(x)\Gamma(x+2\beta)}{\Gamma^2(x+\beta)} \geq 1 + \frac{\beta^2}{x}, \qquad x > 0, x + 2\beta > 0. \tag{13.1}$$

There is the following relationship between T and Gautschi's ratio Q:

$$T(x, x+2\beta) = \frac{Q(x+\beta, \beta)}{Q(x, \beta)}, \tag{13.2}$$

and, naturally, this relationship can be used to produce inequalities of Gurland type from inequalities of Gautshi type. However, techniques explained in previous sections can be used here as well. The following properties of Gurland's ratio are proved in [22]:

Theorem 13.1. *(i) For any $\beta \in I = (0, +\infty)$, the functions*

$$x \mapsto T(x, x+2\beta) \quad \text{and} \quad x \mapsto \log T(x, x+2\beta)$$

are completely monotonic on I.

(ii) For any $\beta \in I$ the function $x \mapsto T(x, x+2\beta)$ is decreasing in $x \in I$ from $+\infty$ to 1.

(iii) For any $x \in I$, the function $\beta \mapsto T(x, x+2\beta)$ is increasing in $\beta \in I$ from 1 to $+\infty$.

(iv) The function $(x, y) \mapsto F(x, y)$ is Schur-convex on $I \times I$, that is, for any $x, y \in I$ such that $x < y$ and $0 < \varepsilon < (y-x)/2$,

$$T(x+\varepsilon, y-\varepsilon) < T(x, y).$$

(v) For any $\beta \in I$, the functions

$$x \mapsto T(x, x+2\beta), \quad x \mapsto \log T(x, x+2\beta), \quad x \mapsto \log\log T(x, x+2\beta)$$

are convex on I.

The initial interest for Gurland's ratio was related to the Cramér-Rao inequality in Statistics. There were many attempts to improve inequality (13.1) by using different versions of the Cramér-Rao inequality. A survey of this early work can be found in [28].

However, G.N. Watson [40] for the case $\beta = 1/2$ and A.V.Boyd [4] for the general case, noticed that (13.1) is a simple consequence of Gauss' formula for the hypergeometric function (see [41, 14.11]):

$$\frac{\Gamma(x)\Gamma(x+2\beta)}{\Gamma^2(x+\beta)} = F(-\beta, -\beta, x, 1) = 1 + \sum_{k=1}^{\infty} \frac{\left((-\beta)_k\right)^2}{k!(x)_k}, \tag{13.3}$$

where $(z)_k = z(z+1)\cdots(z+k-1)$ and F is the hypergeometric function. The series is convergent whenever $x + 2\beta > 0$. If, in addition, $x > 0$, then all terms are nonnegative and, by retaining a finite number of terms in the series, we get (13.1) and its improvements.

We will give here some examples of inequalities of Gurland type.

From the convexity of the function F_2 in Section 11, and Jensen's inequality

$$F_2\left(\frac{x+y}{2}\right) \le \frac{F_2(x) + F_2(y)}{2},$$

we get the inequality [16]:

$$T(x,y) \geq \frac{x^x y^y}{\left(\frac{x+y}{2}\right)^{x+y}}, \tag{13.4}$$

which can be turned into an expansion using a π_n-transform:

$$T(x,y) \gtrsim \frac{(x+n)^{x+n}(y+n)^{y+n}}{\left(\frac{x+y}{2}+n\right)^{x+y+2n}} \rho(x,y,n).$$

A general method of producing double inequalities of Gurland type is the following. Suppose that the function

$$F(x) = \log Q(x,\beta) + \log D(x,\beta), \tag{13.5}$$

is convex with respect to $x \in I$, for a fixed $\beta \in (0,1)$. Then from Jensen's inequality

$$F(x) \leq \beta F(x-1+\beta) + (1-\beta)F(x+\beta), \quad x > 1-\beta \tag{13.6}$$

we find, writing for simplicity $Q(x,\beta) = Q(x)$ and $D(x,\beta) = D(x)$:

$$Q(x)D(x) \leq Q^\beta(x-1+\beta)Q^{1-\beta}(x+\beta)D^\beta(x-1+\beta)D^{1-\beta}(x+\beta). \tag{13.7}$$

Now note that

$$\begin{aligned} Q(x-1+\beta) &= \frac{\Gamma(x-1+2\beta)}{\Gamma(x-1+\beta)} \\ &= \frac{x-1+\beta}{x-1+2\beta} \cdot \frac{\Gamma(x+2\beta)}{\Gamma(x+\beta)} = \frac{x-1+\beta}{x-1+2\beta} \cdot Q(x+\beta), \end{aligned}$$

which finally yields, via (13.7) and the relation (13.2),

$$T(x,x+2\beta) \geq \left(\frac{x-1+2\beta}{x-1+\beta}\right)^\beta \cdot \frac{D(x,\beta)}{D^\beta(x-1+\beta,\beta)D^{1-\beta}(x+\beta,\beta)}, \tag{13.8}$$

where $x > 1-\beta$. An upper bound may be found with the same function (13.5), but starting with Jensen's inequality

$$F(x+\beta) \leq (1-\beta)F(x) + \beta F(x+1), \quad x > 0 \tag{13.9}$$

instead of (13.6). In that way, we find

$$T(x,x+2\beta) \leq \left(\frac{x+\beta}{x}\right)^\beta \cdot \frac{D^\beta(x+1,\beta)D^{1-\beta}(x,\beta)}{D(x+\beta,\beta)}, \quad x \in I. \tag{13.10}$$

The same procedure can be applied if F is concave, with $\leq$ and $\geq$ being interchanged.

It is easy to see that the function $x \mapsto Q(x,\beta)$ is log-concave; therefore, in light of previous sections, the function D should be log-convex.

The simplest double inequality of this type is obtained with $F(x) = \log Q(x,\beta)$:

$$\left(1+\frac{\beta}{x}\right)^{\beta} \le T(x,x+2\beta) \le \left(1+\frac{\beta}{x-1+\beta}\right)^{\beta},$$

and the corresponding expansions via π_n-transform:

$$\rho(x,x+2\beta,n)\cdot\left(1+\frac{\beta}{x+n}\right)^{\beta} \lesssim T(x,x+2\beta)$$
$$\lesssim \rho(x,x+2\beta,n)\cdot\left(1+\frac{\beta}{x+n-1+\beta}\right)^{\beta},$$

where $x > 1-\beta$ and $0 \le \beta \le 1$.

Inequalities for the Trigamma function can yield convex or concave functions related to the Gamma function; this method works well for both types of inequalities. For example, it is proved in [22] that, for $x > 0$:

$$\frac{1}{x+1}+\frac{1}{x^2}+\frac{1}{2(x+1)^2} < \Psi'(x) < \frac{1}{x+1}+\frac{1}{x^2}+\frac{1}{(x+1)^2}. \tag{13.11}$$

These inequalities imply that, for $x > 0$, the function $x \mapsto \log\Gamma(x+1) - x\log(x+1)$ is concave, and the function $x \mapsto \log\Gamma(x+1) - (x+1/2)\log(x+1)$ is convex.

For a more complete survey of Gurland type inequalities and their applictions, see [22].

References

[1] M. ABRAMOWITZ and I.A. STEGUN, *A Handbook of Mathematical Functions*, New York, 1965.

[2] E. ARTIN, *The Gamma Function*, Holt, Rinehart and Winston, New York 1964, translation from the German original of 1931.

[3] H. BOHR and J. MOLLERUP, *Laerbog i matematisk Analyse, III*, Kopenhagen 1922.

[4] A.V. BOYD, Gurland's inequality for the Gamma function, *Skand. Aktuarietidiskr.*, **43** (1961), 134–135.

[5] A.V. BOYD, A note on a paper by Uppuluri, *Pacific J. Math.*, **22** (1967), 9–10.

[6] J.T. CHU, A modified Wallis product and some applications, *Amer. Math.Monthly,* **69**(5) (1962), 402–404.

[7] P.J. DAVIS, Leonhard Euler's integral: A historical profile of the Gamma function, *Amer. Math. Monthly*, **66** (1959), 849–869.

[8] A.M. FINK, Kolmogorov-Landau inequalities for monotone functions, *J. Math. Anal. Appl.*, **90** (1982), 251–258.

[9] W. GAUTSCHI, Some elementary inequalities relating to the gamma and incomplete gamma function, *J. Math. and Phys.*, **38** (1959), 77–81.

[10] J. GURLAND, An inequality satisfied by the Gamma function, *Skand. Aktuarietidiskr.,* **39** (1956), 171–172.

[11] G.H. HARDY, J.E. LITTLEWOOD and G. PÓLYA, *Inequalities*, Cambridge University Press, 1st edition 1934, 2nd edition 1952.

[12] H. HARUKI, A new characterization of Euler's gamma function by a functional equation, *Aequationes Math.*, **31**(2-3) (1986), 173–183.

[13] O. HÖLDER, Über die Eigenschaft der Gamma Funktion keiner algebraische Differentialgleichung zu genügen, *Math. Ann.*, **28** (1887), 1–13.

[14] M.E.H. ISMAIL, L. LORCH and M.E. MULDOON, Completely monotonic functions associated with the Gamma function and its q-analogues, *J. Math. Anal. Appl.*, **116** (1986), 1–9.

[15] H.H. KAIRIES, Über die logarithmische Ableitung der Gammafunktion, *Math. Ann.*, **184** (1970), 157–162.

[16] J.D. KEČKIĆ and P.M. VASIĆ, Some inequalities for the Gamma function, *Publ. Inst. Math. Beograd. N. Ser.*, **11(25)** (1971), 107–114.

[17] D. KERSHAW, Some extensions of W. Gautschi's inequalities for the gamma function, *Math. Comp.*, **41** (1983), 607–611.

[18] W. KRULL, Bemerkungen zur Differenzengleichung $g(x+1)-g(x) = \varphi(x)$, *Math. Nachrichten,* **1** (1948), 365–376.

[19] M. KUCZMA, *Functional Equations in a Single Variable*, Polish Scientific Publishers, Warszawa 1968.

[20] I. LAZAREVIĆ and A LUPAŞ, Functional equations for Wallis and Gamma functions, *Univ. Beograd, Publ. Elektrotehn.Fak.Ser.Mat.Fiz.* $\mathrm{N}^0$461-497 (1979), 245-251.

[21] A. MARSHALL and I. OLKIN, *Inequalities: Theory of Majorization and Its Applications,* Academic Press, New York, 1979.

[22] M. MERKLE, Gurland's ratio for the Gamma function, *Computers and Math. with Applications*, **49** (2005), 389-406.

[23] M. MERKLE, Representation of the error term in Jensen's and some related inequalities with applications, *J. Math. Analysis Appl.*, **231** (1999), 76–90.

[24] M. MERKLE, Conditions for convexity of a derivative and some applications to the Gamma function, *Aequ. Math.*, **55** (1998), 273–280.

[25] M. MERKLE, Logarithmic convexity and inequalities for the Gamma function, *J. Math. Analysis Appl.*, **203** (1996), 369–380.

[26] M. MERKLE, Logarithmic concavity of distribution functions, *Proc. Internat. Memorial Conference "D.S. Mitrinović"*, Niš, 1996. Collection: G.V. Milovanović (ed.), *Recent Progress in Inequalities*, Kluwer Academic Publishers, Dordrecht, 1998, 481–484.

[27] M. MERKLE, Convexity, Schur-convexity and bounds for the Gamma function involving the Digamma function, *Rocky Mountain J. Math.*, **28**(3) (1998), 1053–1066.

[28] D.S. MITRINOVIĆ, *Analytic Inequalities*, Berlin-Heidelberg-New York, 1970.

[29] T. POPOVICIU, *Les fonctions convexes*, Actualités Sci. Indust. 992, Paris 1944.

[30] B.R. RAO, On a generalization of Gautschi's inequality, *Skand. Aktuarietidiskr.* 1970, 10–14.

[31] A.W. ROBERTS and D.E. VARBERG, *Convex Functions*, Academic Press, New York, 1973.

[32] R.T. ROCKAFELLAR, *Convex Analysis*, Princeton Math. Ser., No 28, Princeton University Press, Princeton, New Jersey, 1970.

[33] A.L. RUBEL, A survey of transcendentally transcendental functions, *Amer. Math. Monthly*, **96**(9) (1989), 777–788.

[34] I. SCHUR, Über eine Klasse von Mittelbildungen mit Anwendungen die Determinanten, *Theorie Sitzungsber. Berlin. Math. Gesellschaft,* **22** (1923), 9–20; also in: *Issai Schur Collected Works*, A. Brauer and H. Rohrbach eds., Vol. II, pp 416-427, Springer Verlag, Berlin (1973).

[35] D.N. SHANBHAG, On some inequalites satisfied by the gamma function, *Skand. Aktuarietidiskr.* (1964).

[36] J. STOER and C. WITZGALL, *Convexity and Optimization in Finite Dimensions*, Vol.1, Springer Verlag, 1970.

[37] M. TOMIĆ, Théorème de Gauss relatif au centre de gravité et son application, *Bull. Soc. Math. Phys. Serbie*, **1** (1949), 31–40.

[38] J.G. WENDEL, Note on the gamma function, *Amer. Math. Monthly*, **55** (1948), 653-564.

[39] H. WEYL, Inequalities between two kinds of eigenvalues of a linear transformation, *Proc. Nat. Acad. Sci. USA,* **35** (1949), 408–411.

[40] G.N. WATSON, A note on Gamma function, *Proc. Edin. Math, Soc.,* **11** (1959), Notes, 7-9.

[41] E.T. WHITTAKER and G.N. WATSON, *A Course of Modern Analysis*, part II, Chapter 12, 235-264, Cambridge University Press, Fourth Edition, Cambridge 1962.

[42] D.V. WIDDER, *The Laplace Transform*, Princeton University Press, Princeton 1941.

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