

Convexity in the Theory of the Gamma Function

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ABSTRACT

Convexity is a fundamental property of the Gamma function, as shown by pioneering work of Emil Artin, Wolfgang Krull and others. We start with revisiting Krull's work about the functional equation $f(x+1) - f(x) = g(x)$, in a more modern presentation and a slightly more general setup. We present applications of these results to deriving classical and new expansions, representations and characterizations of the Gamma function. **Keywords:** Functional equation, Gamma function, convexity. **2000 Mathematics Subject Classification:** 26A51, 33B22, 33B15.

1 Introduction

Leonhard Euler described what we call today the Gamma function, in two letters to German mathematician Christian Goldbach in 1729–1730. He published his discovery in (Euler, 1738), which can be found in Internet, translated from Latin to English by Stacy G. Langton in 1999. In fact, Euler discovered two representations of a function $x \mapsto f(x)$ that for $x = n \in \mathbb{N}$ takes value of $n!$. One is infinite product, and the other is integral. Euler notices that the infinite product

$$\prod_{k=1}^{+\infty} \frac{k^{1-x}(k+1)^x}{x+k} \quad (1.1)$$

takes value $x!$ when $x \in \mathbb{N}$. He states that he found a general expression to describe a "progression"

$$1, 1 \cdot 2, 1 \cdot 2 \cdot 3, 1 \cdot 2 \cdot 3 \cdot 4, \dots$$

and he observes that the formula (1.1) is suitable "for interpolating terms whose indices are fractional numbers". In the same paper, Euler derives the integral form of (1.1):

$$\int_0^1 (-\log x)^n dx.$$

Today's familiar form

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt, \quad \Gamma(n) = (n-1)!, \quad n \in \mathbb{N}, \tag{1.2}$$

is due to Legendre. Many celebrated mathematicians contributed to the theory of the Gamma function; a more detailed historical account can be found in (Davis, 1959). This paper is concentrated on convexity: it turns out that this simple tool can be used to produce many interesting properties, inequalities and expansions related to the Gamma function of a positive argument. Immediately from (1.2), we can conclude that $x \mapsto \Gamma(x)$ is convex on $x > 0$, because the function $x \mapsto t^{x-1}$ is convex for each $t > 0$, and $e^{-t} > 0$. By a more subtle argument, that can be found in (Artin, 1964), it can be shown that the sum of log-convex functions (i.e., functions whose logarithm is convex) is also log-convex; hence the Gamma function is also log-convex, because the function $x \mapsto \log t^{x-1}$ is convex for each $t > 0$. In the first half of 20th century, convexity was a contemporary topic in mathematics. The interest in the subject arose with Jensen's papers (Jensen, 1905) and (Jensen, 1906), which prompted mathematical community to search various applications of convexity. Bohr and Mollerup (Bohr and Mollerup, 1922) in 1922 gave a characterization of the Gamma function via convexity. Emil Artin in his tiny monograph (Artin, 1964) of 1931 and Krull in the paper (Krull, 1948) of 1948 extended and ramified the original ideas. While Artin's work is well known, Krull's results are rarely cited in the literature, with an exception of a brief note in Kuczma's monograph (Kuczma, 1968). We start with revisiting Krull's work in Sections 2 and 3, in a more modern presentation and slightly more general setup. In Section 4, we present applications to the Gamma function, and Sections 5 and 6 are based on further applications of convexity, along the lines of papers (Merkle, 2005)-(Merkle, 1996).

2 Krull's work revisited: some auxilliary results

In this section we will repeatedly make use of the following characterization of convexity (Marshall and Olkin, 1979, 16B.3.a): A function f is convex on an interval I if and only if

$$\frac{f(y_1) - f(x_1)}{y_1 - x_1} \leq \frac{f(y_2) - f(x_2)}{y_2 - x_2}, \tag{2.1}$$

whenever $x_1 < y_1 \leq y_2$ and $x_1 \leq x_2 < y_2$, for all $x_1, x_2, y_1, y_2 \in I$.

Lemma 2.1. *Let f be a convex function on $(a, +\infty)$ and suppose that $\lim_{x \rightarrow +\infty} (f(x+1) - f(x)) = 0$. Then*

$$\lim_{x \rightarrow +\infty} (f(x+y) - f(x)) = 0 \quad \text{for every } y \in \mathbb{R}. \tag{2.2}$$

Proof. We first prove the result for $y = h \in (0, 1]$. Let $\varepsilon > 0$ be given. Then there is an $x_0 > a$ so that $-\varepsilon < f(x+1) - f(x) < \varepsilon$ for every $x \geq x_0$. Let now $x \geq x_0 + 1$. Then by convexity of f we have

$$-\varepsilon < f(x) - f(x-1) \leq \frac{f(x+h) - f(x)}{h} \leq f(x+1) - f(x) < \varepsilon, \quad \text{for } h \in (0, 1],$$

wherefrom it follows that $|f(x+h) - f(x)| < \varepsilon h$ for any $x \geq x_0 + 1$. Therefore,

$$\lim_{x \rightarrow +\infty} (f(x+h) - f(x)) = 0 \quad \text{for any } h \in [0, 1]. \tag{2.3}$$

Let now $y > 1$ be fixed, and let $[y] = m$, so that $y = m + h$, $h \in [0, 1)$. Then

$$\begin{aligned} f(x+y) - f(x) &= (f(x+m+h) - f(x+m)) \\ &\quad + (f(x+m) - f(x+m-1)) + \dots + (f(x+1) - f(x)) \end{aligned}$$

and by (2.3) and the assumption we get (2.2) for $y \geq 0$. The statement for $y < 0$ follows from the symmetry. □

Lemma 2.2. *Let f be a convex function on $(a, +\infty)$ and suppose that $\lim_{x \rightarrow +\infty} (f(x+1) - f(x)) = 0$. Then f is non-increasing on $(a, +\infty)$.*

Proof. By convexity, for each $x > a$, $h > 0$ and $n > 0$ we have

$$f(x+h) - f(x) \leq f(x+n+h) - f(x+n).$$

Letting here $n \rightarrow +\infty$ and using Lemma 2.1, we get that $f(x+h) - f(x) \leq 0$ for any $x > a$, $h > 0$. □

Lemma 2.3. *Let f be a non-increasing convex function on $(a, +\infty)$. Then for every $x_0 \geq a$ and $x \in (x_0, x_0 + 1]$, the series*

$$\sum_{k=0}^{+\infty} \left(\frac{f(x+k) - f(x_0+k)}{x-x_0} - (f(x_0+k+1) - f(x_0+k)) \right) \tag{2.4}$$

is convergent.

Proof. Let S_n be a partial sum of the series in (2.4). Then, by convexity criterion (2.1),

$$S_{n+1} - S_n = \frac{f(x+n+1) - f(x_0+n+1)}{x-x_0} - (f(x_0+n+2) - f(x_0+n+1)) \leq 0,$$

and, further,

$$\begin{aligned} S_n &\geq \frac{f(x) - f(x_0)}{x-x_0} - (f(x_0+1) - f(x_0)) \\ &\quad + \sum_{k=1}^n ((f(x_0+k) - f(x_0+k-1)) - (f(x_0+k+1) - f(x_0+k))) \\ &= \frac{f(x) - f(x_0)}{x-x_0} + f(x_0+n) - f(x_0+n+1) \\ &\geq \frac{f(x) - f(x_0)}{x-x_0}. \end{aligned}$$

Hence, the sequence $\{S_n\}$ is convergent, being non-increasing and bounded from below. □

Lemma 2.4. *The series (2.4) is convergent for all $x \neq x_0$.*

Proof. Let $S_n(x)$ be the n -th partial sum of the series (2.4) and let $\varphi_n(x) = (x - x_0)S_n(x)$ for $x \neq x_0$ and $\varphi_n(x_0) = 0$. Then

$$\varphi_n(x + 1) = \varphi_n(x) + f(x + n + 1) - f(x_0 + n + 1) + f(x_0) - f(x). \tag{2.5}$$

By Lemma 2.1, $f(x + n + 1) - f(x_0 + n + 1) \rightarrow 0$ as $n \rightarrow +\infty$ and so $\varphi_n(x + 1)$ is convergent if and only if $\varphi_n(x)$ is. Therefore, if $\varphi_n(x)$ converges in an interval of a length one, then it converges for all $x \in \mathbb{R}$. Note that (2.5) reveals that $\varphi_n(x_0 + 1) = \varphi_n(x_0) = 0$. \square

Lemma 2.5. *Let f be a convex function on $(a, +\infty)$ and suppose that $\lim_{x \rightarrow +\infty} (f(x + 1) - f(x)) = 0$. Then for every $x > a$, $h_1, h_2 \in (0, 1)$ such that $x - h_1 > a$, the series*

$$\sum_{k=0}^{+\infty} \left(\frac{f(x + k + h_2) - f(x + k)}{h_2} - \frac{f(x + k) - f(x + k - h_1)}{h_1} \right) \tag{2.6}$$

converges and for its sum $S(x)$ we have

$$\lim_{x \rightarrow +\infty} S(x) = 0. \tag{2.7}$$

Proof. By convexity, all terms of the series in (2.6) are non-negative. Also by convexity,

$$\frac{f(x + k + h_2) - f(x + k)}{h_2} \leq \frac{f(x + k + 1) - f(x + k + 1 - h_1)}{h_1}.$$

Denoting by D_{k+1} the right hand side above and by S_n the n -th partial sum of the series (2.6), we have that

$$\begin{aligned} 0 \leq S_n &\leq \sum_{k=0}^n (D_{k+1} - D_k) = D_{n+1} - D_0 \\ &= \frac{f(x + n + 1) - f(x + n + 1 - h_1)}{h_1} - \frac{f(x) - f(x - h_1)}{h_1}. \end{aligned}$$

Now, by Lemma 2.2, $f(x + n + 1) \leq f(x + n + 1 - h_1)$ and $f(x) \leq f(x - h_1)$ for $x - h_1 > a$, so

$$0 \leq S_n \leq \frac{f(x - h_1) - f(x)}{h_1} \rightarrow 0 \quad \text{as } x \rightarrow +\infty. \tag{2.8}$$

This shows that the series in (2.6) is convergent and that its sum converges to 0 as $x \rightarrow +\infty$. \square

Lemma 2.6. *Suppose that $\rho(x + 1) = \rho(x)$ for every $x > a$ and suppose that*

$$\frac{\rho(x + h_2) - \rho(x)}{h_2} - \frac{\rho(x) - \rho(x - h_1)}{h_1} \rightarrow 0 \quad \text{as } x \rightarrow +\infty, \tag{2.9}$$

for all $h_1, h_2 < \delta$, for some $\delta > 0$. Then the function $x \mapsto \rho(x)$, $x > a$, is identically equal to a constant.

Proof. For any fixed $x_0 > a$, choose $h_1 < \delta$ so that $x_0 - h_1 > a$ and take arbitrary $h_2 < \delta$. Then by the assumptions we have that

$$\begin{aligned} &\frac{\rho(x_0 + h_2) - \rho(x_0)}{h_2} - \frac{\rho(x_0) - \rho(x_0 - h_1)}{h_1} \\ &= \frac{\rho(x_0 + n + h_2) - \rho(x_0 + n)}{h_2} - \frac{\rho(x_0 + n) - \rho(x_0 + n - h_1)}{h_1} \rightarrow 0 \end{aligned}$$

as $n \rightarrow +\infty$, and we conclude that

$$\frac{\rho(x_0 + h_2) - \rho(x_0)}{h_2} - \frac{\rho(x_0) - \rho(x_0 - h_1)}{h_1} = 0$$

for any h_1, h_2 as specified above. Then ρ is both convex and concave, hence $\rho(x) = bx + c$ for $x \in [x_0 - h_1, x_0 + h_2]$. Since x_0 was arbitrary we must have $\rho(x) = bx + c$ for some constants b, c and all $x > a$; since $\rho(x + 1) = \rho(x)$, we conclude that $b = 0$ and so, ρ is a constant. \square

Lemma 2.7. Lemmas 2.1, 2.3, 2.4 and 2.5 hold true if conditions on f are replaced by $f(x) = f_1(x) + f_2(x)$, where f_1 is convex and f_2 is concave on $(a, +\infty)$ and $\lim_{x \rightarrow +\infty} (f_i(x+1) - f_i(x)) = 0, i = 1, 2$.

Proof. It is easy to see that the domain of validity of mentioned lemmas is linear, i.e., if the stated results hold for some functions f and g , they hold for any linear combination $\alpha f + \beta g$. In particular, since the results are proved for convex functions f , they must hold for concave functions $-f$ and then for the sum of a convex and a concave function. \square

3 Krull’s work revisited: main results

Let us observe the functional equation

$$g(x + 1) - g(x) = f(x), \quad x \geq a, \tag{3.1}$$

where g is an unknown function and f is given. Note that the case $f(x) = \log x$ is the recurrence relation for the logarithm of the Gamma function. In what follows we will make use of the following two conditions. **Condition A.** We say that f satisfies condition A if $f(x) = f_1(x) + f_2(x)$, where f_1 is convex and f_2 is concave on $x > a$, and also

$$\lim_{x \rightarrow +\infty} (f_i(x + 1) - f_i(x)) = 0, \quad i = 1, 2.$$

Condition B. A function g satisfies condition B if for any h_1, h_2 such that $0 < h_i < \delta$ for some $\delta > 0$, we have that

$$\lim_{x \rightarrow +\infty} \left(\frac{g(x + h_2) - g(x)}{h_2} - \frac{g(x) - g(x - h_1)}{h_1} \right) = 0.$$

Remark 3.1. Let $\varphi(x, t)$ be a function of two real variables, such that $x \mapsto \varphi(x, t)$ is convex on $x > a$ for each $t \in \mathbb{R}$, and let μ be a finite signed measure on \mathbb{R} , such that the function $t \mapsto \varphi(x, t)$ is μ -integrable for each x . Assume, further, that

$$\lim_{x \rightarrow +\infty} (\varphi(x + 1, t) - \varphi(x, t)) = 0 \quad \text{uniformly in } t \in \mathbb{R}. \tag{3.2}$$

Then the function f defined by

$$f(x) = \int_{\mathbb{R}} \varphi(x, t) \, d\mu(t) \tag{3.3}$$

satisfies condition A. Indeed, by Jordan decomposition, $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are positive finite measures, and we may define

$$f_1(x) = \int_{\mathbb{R}} \varphi(x, t) \, d\mu^+(t), \quad f_2(x) = - \int_{\mathbb{R}} \varphi(x, t) \, d\mu^-(t).$$

Then $f(x) = f_1(x) + f_2(x)$, where f_1 is convex and f_2 is concave; the rest follows from (3.2). In particular, condition A is satisfied by every function of the form

$$f(x) = \int_{-\infty}^{+\infty} K(t)\varphi(x, t) \, dt,$$

where K is an integrable function, and $x \mapsto \varphi(x, t)$ is convex and satisfies (3.2).

Remark 3.2. Note that the condition B is satisfied with any twice differentiable function g such that $g''(x) \rightarrow 0$ as $x \rightarrow +\infty$.

Lemma 3.1. *Suppose that the equation (3.1) has a solution g that satisfies condition B. Then all solutions that satisfy condition B are of the form $g + C$, where C is an arbitrary constant.*

Proof. Suppose that g_1 is another solution of (3.1) that satisfies condition B and let $\rho(x) = g_1(x) - g(x)$. Then by Lemma 2.6, ρ is a constant, which had to be proved. □

Lemma 3.2. *Suppose that f satisfies the condition A, and let g be a convex solution of (3.1). Then g satisfies the condition B. Moreover, any solution of (3.1) that satisfies the condition B is of the form $g + C$, where C is an arbitrary constant. In particular, any convex solution of (3.1) is of the form $g + C$ and any twice differentiable solution with a second derivative converging to zero as $x \rightarrow +\infty$ is also of the form $g + C$.*

Proof. Let f satisfies the condition A, and let g be a convex solution of (3.1). Then for any $h_1, h_2 \in (0, 1)$ we have that

$$\begin{aligned} f(x+1) - f(x) &= (g(x+2) - g(x+1)) - (g(x+1) - g(x)) \\ &\geq \frac{g(x+1+h_2) - g(x+1)}{h_2} - \frac{g(x+1) - g(x+1-h_1)}{h_1} \geq 0 \end{aligned}$$

By condition A, $f(x+1) - f(x) \rightarrow 0$ as $x \rightarrow +\infty$ and so we conclude that g satisfies the condition B. Then by Lemma 3.1, all solutions of (3.1) that satisfy the condition B are of the form $g + C$. As we just proved, any convex solution of (3.1) satisfies the condition B, and so, any convex solution is of the form $g + C$. Finally, as we noticed in Remark 3.2, any function with the second derivative converging to zero as $x \rightarrow +\infty$ satisfies condition B and hence any such a solution is of the form $g + C$. □

Theorem 3.3. *Suppose that f satisfies condition A. For $x_0 \geq a$ being fixed, define a function g by*

$$\begin{aligned} g(x) &= \int_{x_0}^x f(u) \, du - \frac{1}{2}f(x) \\ &+ \sum_{k=0}^{+\infty} \left(\int_{x+k}^{x+k+1} f(u) \, du - \frac{1}{2}(f(x+k+1) + f(x+k)) \right), \end{aligned} \tag{3.4}$$

where $x > a$. Then (i) The function g is a solution of equation (3.1) on $x > a$. (ii) The function g satisfies condition B. (iii) The following relation holds:

$$g(x) = \int_{x_0}^x f(u) \, du - \frac{1}{2}f(x) + o(1) \quad (x \rightarrow +\infty). \tag{3.5}$$

(iv) In addition, if f is either convex or concave, then

$$|S(x)| \leq \frac{1}{2} |f(x + 1/2) - f(x)|, \tag{3.6}$$

where $S(x)$ is the sum of the series in (3.4).

Proof. It is clear that if the theorem holds in the case of convex f satisfying condition A, then it must hold for any function that satisfies condition A. So, it suffices to assume that f is a convex function that satisfies condition A. We firstly have to prove that the series in (3.4) converges. Let us introduce the notation

$$D(x) = \int_x^{x+1} f(u) \, du - \frac{1}{2}(f(x + 1) + f(x)).$$

Then by Hadamard's inequalities,

$$f(x + 1/2) \leq \int_x^{x+1} f(u) \, du \leq \frac{1}{2}(f(x) + f(x + 1))$$

and therefore

$$\begin{aligned} 0 &\geq D(x + k) \geq f(x + k + 1/2) - \frac{1}{2}(f(x + k + 1) + f(x + k)) \\ &= -\frac{1}{4} \left(\frac{f(x + k + 1) - f(x + k + 1/2)}{1/2} - \frac{f(x + k + 1/2) - f(x + k)}{1/2} \right). \end{aligned} \tag{3.7}$$

Denote by $h(x + k)$ the expression on the right hand side of (3.7). By Lemma 2.5, applied with $h_1 = h_2 = 1/2$ and with x replaced by $x + 1/2$, the series $\sum h(x + k)$ converges for any $x > a$ and its sum converges to zero as $x \rightarrow +\infty$. By (3.7), the same holds true for the series $\sum D(x + k)$. This proves (iii) in the case of convex f . By linearity, (iii) holds for any function f that satisfies condition A. If f is convex, (iv) follows from (3.7) and (2.8); in a similar way one can obtain (iv) for a concave f . Note that (iv) need not hold under general setup of condition A, as it was incorrectly deduced in Krull's original paper. Let us now prove that g satisfies the condition B. By (iii) and Lemma 2.5, it suffices to show that the condition B is satisfied by the function ψ defined by $\psi(x) = \int_{x_0}^x f(u) \, du$. For any $h_1, h_2 > 0$ we have

$$\begin{aligned} \frac{\psi(x + h_2) - \psi(x)}{h_2} - \frac{\psi(x) - \psi(x - h_1)}{h_1} &= \frac{1}{h_2} \int_x^{x+h_2} f(u) \, du - \frac{1}{h_1} \int_{x-h_1}^x f(u) \, du \\ &= f(x_2) - f(x_1), \end{aligned}$$

where $x_1 \in (x - h_1, x)$ and $x_2 \in (x, x + h_2)$ (by the mean value theorem and continuity of a convex function in an open interval). By Lemma 2.2, f is non-increasing and therefore,

$$0 \geq f(x_2) - f(x_1) \geq f(x + h_2) - f(x - h_1) \rightarrow 0 \quad \text{as } x \rightarrow +\infty \quad (\text{by Lemma 2.1}).$$

Therefore, the condition B holds for g . Finally, let us verify that g is a solution of (3.1). By (3.4) we have that $g(x) = \lim_{n \rightarrow +\infty} g_n(x)$, where

$$g_n(x) = \int_{x_0}^{x+n+1} f(u) \, du - \sum_{k=0}^{n+1} f(x+k) + \frac{1}{2}f(x+n+1),$$

and

$$g_n(x+1) - g_n(x) = \int_{x+n+1}^{x+n+2} f(u) \, du - f(x+n+2) + f(x) + \frac{1}{2}(f(x+n+2) - f(x+n+1))$$

Now by Lemma 2.1,

$$\lim_{n \rightarrow +\infty} (f(x+n+2) - f(x+n+1)) = 0.$$

If f is convex then by Hadamard's inequalities,

$$f(x+n+1/2) - f(x+n+2) \leq \int_{x+n+1}^{x+n+2} f(u) \, du - f(x+n+2) \leq \frac{1}{2}(f(x+n+1) - f(x+n+2))$$

and so

$$\lim_{n \rightarrow +\infty} \left(\int_{x+n+1}^{x+n+2} f(u) \, du - f(x+n+2) \right) = 0.$$

By linearity, the last relation holds also if f is the sum of a convex and a concave function. Therefore, we have shown that

$$g(x+1) - g(x) = \lim_{n \rightarrow +\infty} (g_n(x+1) - g_n(x)) = f(x),$$

i.e., the equation (3.1) is satisfied. □

Theorem 3.4. Suppose that f satisfies condition A. For $x_0 \geq a$ and $y_0 \in \mathbb{R}$ being fixed, define a function g^* by

$$g^*(x) = y_0 + (x - x_0)f(x_0) - \sum_{k=0}^{+\infty} (f(x+k) - f(x_0+k)) - (x - x_0)(f(x_0+k+1) - f(x_0+k)). \quad (3.8)$$

Then (i) The function g^* satisfies (3.1) on $x > a$ and $g^*(x_0) = y_0$. (ii) The function g^* satisfies the condition B.

Proof. The series in (3.8) converges by Lemma 2.4. Now it is easy to verify (i). Condition B follows from Lemma 2.5. □

Corollary 3.5. Let g and g^* be as defined in Theorems 3.3 and 3.4 respectively. Then for $x > a$, $g(x) = g^*(x) + C$, where C is a constant.

Proof. By Lemma 3.1 and Theorems 3.3 and 3.4. □

Corollary 3.6. If f is a concave function satisfying the condition A, then the function g^* is a unique convex solution of (3.1) under the initial condition $g(x_0) = y_0$.

Proof. We only have to show that g^* is convex. Then the rest follows from Lemma 3.2. However, from (3.8) it follows easily that

$$\begin{aligned} &g(\lambda x + (1 - \lambda)y) - \lambda g(x) - (1 - \lambda)g(y) \\ &= - \sum_{k=0}^{+\infty} f(\lambda(x+k) + (1-\lambda)(y+k)) - \lambda f(x+k) - (1-\lambda)f(y+k), \end{aligned}$$

and therefore, concavity of f implies the convexity of g . □

Theorem 3.7. *Suppose that f satisfies the condition A and assume also that f is r times differentiable, with $f^{(r)}(x)$ monotone for x large enough. Then the solution g of (3.1), introduced in Theorems 3.3 and 3.4 is also r times differentiable and we have*

$$g^{(j)}(x) = - \sum_{k=0}^{+\infty} f^{(j)}(x+k) \quad (j \geq 2) \tag{3.9}$$

$$g'(x) = \lim_{n \rightarrow +\infty} \left(f(x+n) - \sum_{k=0}^n f'(x+k) \right). \tag{3.10}$$

Proof. By repeated formal differentiation in (3.4) we obtain

$$\begin{aligned} g^{(j)}(x) &= f^{(j-1)}(x) - \frac{1}{2}f^{(j)}(x) \\ &+ \sum_{k=0}^{+\infty} \left(\int_{x+k}^{x+k+1} f^{(j)}(u) \, du - \frac{1}{2}(f^{(j)}(x+k+1) + f^{(j)}(x+k)) \right) \end{aligned} \tag{3.11}$$

Let us show that the term-wise differentiation is allowed. Firstly, note that if f is not monotone in any interval $[b, +\infty)$, then its derivative has infinitely many changes of sign, hence it can not be monotone. Therefore, if f' is monotone for x large enough, then so is f , and hence the monotonicity of $f^{(r)}$ for x large enough implies the same property for $f, f', \dots, f^{(r-1)}$. Next, observe that if $x_2 = x_1 + 1$ and if φ is any integrable and monotone function, say monotonically nondecreasing, then

$$\varphi(x_1) \leq \int_{x_1}^{x_2} \varphi(u) \, du \leq \varphi(x_2),$$

and opposite inequalities hold if φ is nonincreasing; in a general case we have

$$\left| \int_{x_1}^{x_2} \varphi(u) \, du - \frac{1}{2}(\varphi(x_1) + \varphi(x_2)) \right| \leq \frac{1}{2}|\varphi(x_2) - \varphi(x_1)|.$$

An application to the sum in (3.11) yields

$$\begin{aligned} &\left| \sum_{k=n+1}^{n+m} \int_{x+k}^{x+k+1} f^{(j)}(u) \, du - \frac{1}{2}(f^{(j)}(x+k+1) + f^{(j)}(x+k)) \right| \\ &\leq \sum_{k=n+1}^{n+m} \frac{1}{2}|f^{(j)}(x+k+1) - f^{(j)}(x+k)| \\ &= \frac{1}{2}|f^{(j)}(x+n+m+1) - f^{(j)}(x+n+1)| \quad (\text{by monotonicity}), \end{aligned} \tag{3.12}$$

where we assumed (without a loss of generality) that $f^{(r)}(x)$ is monotone for $x > a$. Now the monotonicity of the first derivative implies

$$|f(x+1) - f(x)| = |f'(x+\theta)| \geq |f'(x)| \quad \text{or}$$

$$|f(x + 1) - f(x)| = |f'(x + \theta)| \geq |f'(x + 1)|$$

for all x large enough. Then by the condition A we conclude that $f'(x) \rightarrow 0$ as $x \rightarrow +\infty$ and further $\lim_{x \rightarrow +\infty} (f'(x + 1) - f'(x)) = 0$. Repeating the procedure for higher derivatives, we conclude that $\lim_{x \rightarrow +\infty} f^{(j)}(x) = 0$ for all $j = 1, \dots, r$. This, together with (3.12) shows that the series in (3.11) is uniformly convergent and the term-wise differentiation in (3.4) is allowed. Retaining the first n terms in (3.11) we obtain

$$g^{(j)}(x) = \lim_{n \rightarrow +\infty} \left(f^{(j-1)}(x + n + 1) - \frac{1}{2}f^{(j)}(x + n + 1) - \sum_{k=0}^n f^{(j)}(x + k) \right).$$

By the above discussion, the second term above vanishes as $n \rightarrow +\infty$ for all $j \geq 1$ and so does the first term for $j > 1$. For $j = 1$, the first term may not vanish and we get (3.10). \square

Corollary 3.8. *If f satisfies the condition A and if f'' is monotone for large enough x , then*

$$g(x)'' = - \sum_{k=0}^{+\infty} f''(x + k) \quad \text{and} \quad \lim_{x \rightarrow +\infty} g''(x) = 0.$$

Proof. The expression for g'' in terms of the series follows directly from Theorem 3.7. Since f'' is monotone it follows that it does not change sign for x large enough and so, f is either concave or convex. Then by Lemma 2.5 we conclude that $g''(x) \rightarrow 0$ as $x \rightarrow +\infty$. \square

4 Applications to the Gamma function

In this section we present some classical results related to the Gamma function as straightforward consequences of Krull's theory. It seems that the result stated in Theorem 4.2 has not been observed in the literature.

Theorem 4.1 (Bohr-Mollerup theorem). *If G is a logarithmically convex solution of the functional equation*

$$xG(x) = G(x + 1) \quad (x > 0)$$

and if $G(1) = 1$, then $G(x) \equiv \Gamma(x)$ on $x > 0$.

Proof. By Lemma 3.2, with $g(x) = \log G(x)$ and $f(x) = \log x$. \square

Theorem 4.2. *If g is a twice differentiable function that satisfies*

$$g(x + 1) - g(x) = \log x, \quad g(1) = 0, \quad \text{and} \quad \lim_{x \rightarrow +\infty} g''(x) = 0,$$

then $g(x) = \log \Gamma(x)$.

Proof. By Lemma 3.1, with $f(x) = \log x$. \square

Theorem 4.3. *The following representation holds for $x > 0$:*

$$\Gamma(x) = \lim_{n \rightarrow +\infty} \frac{(x + n + 1)^{x+n+1/2} e^{-(x+n+1)} \sqrt{2\pi}}{x(x + 1) \cdots (x + n)}.$$

Proof. Any solution of equation (3.1) can be expressed, by Theorem 1 and Lemma 8, as

$$g(x) = \lim_{n \rightarrow +\infty} \left(F(x+n+1) - \sum_{k=0}^n f(x+k) - \frac{1}{2}f(x+n+1) \right) + C,$$

where C is a constant, F is a primitive function for f . In the case of the Gamma function, we have that $f(x) = \log x, F(x) = x \log x - x, x > 0$ and so

$$g(x) = \lim_{n \rightarrow +\infty} \left((x+n+1) \log(x+n+1) - (x+n+1) - \sum_{k=0}^n \log(x+k) - \frac{1}{2} \log(x+n+1) \right) + C.$$

To determine C , let $x = 1$, then $g(1) = \log \Gamma(1) = 0$ and so

$$\lim_{n \rightarrow +\infty} \left((n+2) \log(n+2) - (n+2) - \sum_{k=1}^{n+1} \log k - \frac{1}{2} \log(n+2) \right) + C = 0.$$

Using Stirling's formula for the factorial, the expression under limit can be transformed as

$$\log \frac{(n+2)^{n+3/2} e^{-(n+2)}}{(n+1)!} \sim \log \frac{(n+2)^{n+3/2} e^{-(n+2)}}{\sqrt{2\pi}(n+1)^{n+3/2} e^{-(n+1)}} \sim -\frac{1}{2} \log 2\pi,$$

and so, $C = \frac{1}{2} \log 2\pi$. This yields the desired expansion. □

Theorem 4.4 (Euler's product). For $x > 0$, the Gamma function can be expressed as

$$\Gamma(x) = \frac{1}{x} \prod_{k=1}^{+\infty} \left(1 + \frac{1}{k} \right)^x \left(1 + \frac{x}{k} \right)^{-1}.$$

Proof. A straightforward application of (3.8), with $f(x) = \log x, x_0 = 1, y_0 = 0$ and with x replaced by $x + 1$ gives

$$\log \Gamma(x+1) = \log \prod_{k=1}^{+\infty} \frac{k^{1-x}(k+1)^x}{(x+k)},$$

which yields the desired result. □

Theorem 4.5. The derivative of the function $x \mapsto \log \Gamma(x)$ can be expressed as

$$\Psi(x) = \frac{d}{dx} \log \Gamma(x) = \lim_{n \rightarrow +\infty} \left(\log(x+n) - \sum_{k=0}^n \frac{1}{x+k} \right) \tag{4.1}$$

$$= -\gamma - \frac{1}{x} + \sum_{k=1}^{+\infty} \frac{x}{k(x+k)}, \tag{4.2}$$

where γ is Euler's constant. Further, for higher derivatives we have

$$\Psi^{(j-1)}(x) = \frac{d^j}{dx^j} \log \Gamma(x) = \sum_{k=0}^{+\infty} \frac{(-1)^j (j-1)!}{(x+k)^j}, \quad j = 2, 3, \dots \tag{4.3}$$

Proof. Expression (4.1) follows from (3.10) of Theorem 3.7, by letting $f(x) = \log x$. Its equivalent form (4.2) is easy to obtain using the definition of Euler's constant,

$$\gamma = \lim_{n \rightarrow +\infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right).$$

The expression (4.3) is obtained from (3.9) with $f(x) = \log x$. □

5 Asymptotic expansions via convexity

We start this section with a result from (Merkle, 1999):

Theorem 5.1. *Let g be a twice continuously differentiable function on an interval $I = (a, +\infty)$. For $\lambda \in [0, 1]$, $x, y \in I$, let*

$$R(\lambda, x, y) = \lambda g(x) + (1 - \lambda)g(y) - g(\lambda x + (1 - \lambda)y).$$

If $\lim_{x \rightarrow +\infty} g''(x) = 0$, then $R(\lambda, x, y) \rightarrow 0$ as $\min(x, y) \rightarrow +\infty$.

Using Theorem 5.1, we will now derive a more informative version of expansion (3.8). In this and next section, we use the notation $a_n \lesssim b_n$ for sequences a_n and b_n with

$$a_n \leq b_n \quad \text{for all } n \quad \text{and} \quad \lim_{n \rightarrow +\infty} (a_n - b_n) = 0.$$

The notation $a_n \gtrsim b_n$ is equivalent to $b_n \lesssim a_n$

Theorem 5.2. *For a given function f , let g be a twice differentiable solution of (3.1) on $x > a$, with the property that $\lim_{x \rightarrow +\infty} g''(x) = 0$. Then, for every $x, y \in (a, +\infty)$, we have*

$$g(y) - g(x) = \lim_{n \rightarrow +\infty} \left((y - x)f(x + n) + \sum_{k=0}^{n-1} (f(x + k) - f(y + k)) \right),$$

$$g(y) - g(x) = \lim_{n \rightarrow +\infty} \left((y - x)f(y + n - 1) + \sum_{k=0}^{n-1} (f(x + k) - f(y + k)) \right).$$

If g is a convex function on $(a, +\infty)$, then for $0 < y - x < 1$, we have

$$g(y) - g(x) \lesssim (y - x)f(x + n) + \sum_{k=0}^{n-1} (f(x + k) - f(y + k)),$$

$$g(y) - g(x) \gtrsim (y - x)f(y + n - 1) + \sum_{k=0}^{n-1} (f(x + k) - f(y + k)),$$

and for $y - x < 0$ or $y - x > 1$:

$$g(y) - g(x) \gtrsim (y - x)f(x + n) + \sum_{k=0}^{n-1} (f(x + k) - f(y + k)),$$

$$g(y) - g(x) \lesssim (y - x)f(y + n - 1) + \sum_{k=0}^{n-1} (f(x + k) - f(y + k)).$$

If g is a concave function, the last four expansions remain valid with \lesssim and \gtrsim interchanged.

Proof. For $0 < y - x < 1$, and an integer $n \geq 1$, we can express $y + n$ as a convex combination of $x + n$ and $x + n + 1$ as follows:

$$y + n = (1 - (y - x))(x + n) + (y - x)(x + n + 1).$$

Then, by Theorem 4.5, $R_n := R(1 - (y - x), x + n, x + n + 1) \rightarrow 0$ as $n \rightarrow +\infty$, where

$$R(1 - (y - x), x + n, x + n + 1) = (1 - (y - x))g(x + n) + (y - x)g(x + n + 1) - g(y + n).$$

Since g satisfies (3.1), we have that

$$g(x + n) = g(x) + \sum_{k=0}^{n-1} f(x + k),$$

and so

$$R_n = g(x) - g(y) + \sum_{k=0}^{n-1} (f(x + k) - f(y + k)) + (y - x)f(x + n + 1),$$

and the first expansion is proved. To prove the second one, we start with a convex combination of $y + n - 1$ and $y + n$

$$x + n = (y - x)(y + n - 1) + (1 - (y - x))(y + n),$$

and proceed in the same way to find that

$$R(y - x, y + n - 1, y + n) = g(y) - g(x) + \sum_{k=0}^{n-1} (f(y + k) - f(x + k)) - (y - x)f(y + n - 1)$$

converges to zero as $n \rightarrow +\infty$. To prove third and fourth relation, note that for a convex function g we have

$$R(1 - (y - x), x + n, x + n + 1) \geq 0 \quad \text{and} \quad R(y - x, y + n - 1, y + n) \geq 0.$$

To get fifth and sixth relation if $y - x > 1$, we start with

$$\begin{aligned} x + n + 1 &= \frac{y - x + 1}{y - x}(x + n) + \frac{1}{y - x}(y + n) \quad \text{and} \\ y + n - 1 &= \frac{1}{y - x}(x + n) + \frac{y - x - 1}{y - x}(y + n), \end{aligned}$$

and if $y - x < 0$ with

$$\begin{aligned} x + n &= \frac{x - y}{1 + x - y}(x + n + 1) + \frac{1}{1 + x - y}(y + n) \quad \text{and} \\ y + n &= \frac{1}{1 + x - y}(x + n) + \frac{x - y}{1 + x - y}(y + n - 1), \end{aligned}$$

and proceed in a same way as above. Finally, the statement for a concave function g follows from the fact that $R \leq 0$ in that case. □

Theorem 5.2 yields various two sided expansions for the Gamma function. For example, letting $x = m$ and $y = m + \beta$, where m is a positive integer, and $\beta \in [0, 1]$, and applying results of Theorem 10 with $f(x) = \log x$, we get

$$\begin{aligned} (m - 1)!(m + n - 1 + \beta)^\beta \prod_{k=0}^{n-1} \frac{m + k}{m + \beta + k} &\lesssim \Gamma(m + \beta) \\ &\lesssim (m - 1)!(m + n)^\beta \prod_{k=0}^{n-1} \frac{m + k}{m + \beta + k}, \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

In the next theorem, we give a refinement of the representation of Theorem 4.3.

Theorem 5.3. For $x > 0$, we have

$$\frac{(x+n)^{x+n-1/2} e^{-(x+n)} \sqrt{2\pi}}{x(x+1)\cdots(x+n-1)} \lesssim \Gamma(x) \lesssim \frac{(x+n)^{x+n-1} \sqrt{x+n+\frac{1}{2}} e^{-(x+n)} \sqrt{2\pi}}{x(x+1)\cdots(x+n-1)}.$$

Proof. Using Theorem 6, we can evaluate the sum of the series in (3.4), with $f(x) = \log x$ and $g(x) = \log \Gamma(x)$, as follows:

$$S(x) = \log \Gamma(x) + x - \left(x - \frac{1}{2}\right) \log x - \frac{1}{2} \log 2\pi. \tag{5.1}$$

By inequality (3.6) for a concave function $f(x) = \log x$, we have that

$$1 \leq e^{S(x)} \leq \sqrt{\frac{x+\frac{1}{2}}{x}}. \tag{5.2}$$

Replacing x with $x+n$ and using the fact that

$$\Gamma(x+n) = x(x+1)\cdots(x+n-1),$$

we get the desired inequalities from (5.2). The asymptotics as $n \rightarrow +\infty$ follows from comparison with the expansion of Theorem 4.3. □

6 Gautchi's and Gurland's ratio

Motivated by various applications, the inequalities related to the Gamma function have been a subject of an intensive research. Apart from inequalities for the Gamma function alone, there is a considerable number of results about two ratios of Gamma functions. Gautchi's ratio (Gautschi, 1959) is defined by

$$Q(x, \beta) = \frac{\Gamma(x+\beta)}{\Gamma(x)}, \tag{6.1}$$

and has been usually studied with $\beta \in (0, 1)$, see (Merkle, 1996) and references therein. Gurland's ratio (Gurland, 1956) is defined as

$$T(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma^2((x+y)/2)}, \quad x, y > 0. \tag{6.2}$$

A survey of results about Gurland's ratio can be found in (Merkle, 2005). Logarithms of both ratios satisfy Krull's functional equation (3.1), with conditions A and B being satisfied:

$$\log Q(x+1, \beta) - \log Q(x, \beta) = \log(x+\beta) - \log x,$$

$$\log T(x+1, x+1+2\beta) - \log T(x, x+2\beta) = \log(x+2\beta) + \log x - 2\log(x+\beta).$$

Therefore, the results and methods presented in previous sections can be applied to produce inequalities and expansions for both ratios. A more detailed account of convexity techniques for inequalities can be found in (Merkle, 2008), as well as in papers (Merkle, 1997), (Merkle, 1998a), (Merkle, 1998b), (Merkle, 2001) and (Merkle, 2004).

References

- Artin, E. 1964. *The Gamma Function*, Holt, Rinehart and Winston, New York. Translation from the German original of 1931.
- Bohr, H. and Mollerup, J. 1922. Laerbog i matematisk analyse iii, Kopenhagen.
- Davis, P. J. 1959. Leonhard Euler's integral: A historical profile of the Gamma function, *Amer. Math. Monthly* **66**: 849–869.
- Euler, L. 1738. De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt (On transcendental progressions that is, those whose general terms cannot be given algebraically), *Commentarii academiae scientiarum Petropolitanae* **5 (1730/1)**: 36–57. In: Opera Omnia I₁₄, 1-24, Leipzig-Berlin 1924; English translation at <http://www.eulerarchive.com>.
- Gautschi, W. 1959. Some elementary inequalities relating to the Gamma and incomplete Gamma function, *J. Math. and Phys.* **38**: 77 – 81.
- Gurland, J. 1956. An inequality satisfied by the Gamma function, *Skand. Aktuarietidiskr.* **39**: 171 – 172.
- Jensen, J. 1905. Om konvekse funktioner og uligheder mellem Middelveerdi, *Nyt. Tidsskrift for Matematik* **16B**: 49-69.
- Jensen, J. 1906. Sur les fonctions convexes et les inégalités entre les valeurs moyennes, *Acta Math.* **30**:175-193.
- Krull, W. 1948. Bemerkungen zur Differenzengleichung, $g(x+1) - g(x) = \varphi(x)$ *Math. Nachrichten* **1**: 365 – 376.
- Kuczma, M. 1968. *Functional equations in a single variable*, Polish Scientific Publishers, Warszawa.
- Marshall, A. and Olkin, I. 1979. *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York.
- Merkle, M. 1996. Logarithmic convexity and inequalities for the gamma function, *J. Math. Analysis Appl.* **203**:369-380.
- Merkle, M. 1997. On log convexity of a ratio of gamma functions, *Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat* **8**: 114 – 119.
- Merkle, M. 1998a. Conditions for convexity of a derivative and some applications to the gamma function, *Aequ. Math.* **55**: 273 – 280.
- Merkle, M. 1998b. Convexity, schur-convexity and bounds for the gamma function involving the digamma function, *Rocky Mountain J. Math.* **28**(3): 1053 – 1066.
- Merkle, M. 1999. Representation of the error term in Jensen's and some related inequalities with applications, *J. Math. Analysis Appl.* **231**: 76 – 90.
- Merkle, M. 2001. Conditions for convexity of a derivative and applications to the gamma and digamma function, *Facta Universitatis (Niš), Ser. Math. Inform.* **16**: 13 – 20.
- Merkle, M. 2004. Reciprocally convex functions, *J. Math. Analysis Appl.* **293**: 210 – 218.
- Merkle, M. 2005. Gurland's ratio for the gamma function, *Computers and Math. with Applications* **49**: 389 – 406.
- Merkle, M. 2008. Inequalities for the Gamma function via convexity, in P. Cerone and S. Dragomir (eds), *Advances in Inequalities for Special Functions*, Nova Science Publishers, pp. 97–120. To appear.