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692. MILLS RATIO FOR THE GAMMA DISTRIBUTION*

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1. The function defined by:

$$R(x)=\frac{1-F(x)}{F'(x)}=\left(\exp\frac{x^2}{2}\right)\int\limits_x^{\infty}\exp\left(-\frac{t^2}{2}\right)\mathrm{d}t \tag{1}$$

is called MILLS ratio [2] for the normal distribution, where $F(x)$ is the distribution function of the $N(0, 1)$ distribution. The computation of this ratio is used, for example, in mathematical statistics and in the theory of difraction. Various approximations for the function (1) are known [1]. Functions with the form (1) have been defined for some other distributions ([3], [4], [5], [6]).

In this paper some approximations for MILLS ratio for the gamma distribution $\Gamma\left(\alpha, \frac{1}{\beta}\right)$ are given.

2. The distribution function for the gamma distribution is defined by:

$$F(x)=\frac{1}{\beta^{\alpha}\,\Gamma(\alpha)}\int\limits_0^{x} t^{\alpha-1}\exp\left(-\frac{t}{\beta}\right)\mathrm{d}t,\qquad x,\,\alpha,\,\beta>0. \tag{2}$$

The MILLS ratio for this distribution is defined by:

$$\mathcal{R}(x)=\frac{1-F(x)}{F(x)}=\beta^{\alpha}\,\Gamma(\alpha)\,\exp\frac{x}{\beta}-x^{1-\alpha}\left(\exp\frac{x}{\beta}\right)\int\limits_0^{x} t^{\alpha-1}\exp\left(-\frac{t}{\beta}\right)\mathrm{d}t. \tag{3}$$

There are satisfied conditions for developing the integral in (3) into power series, so we have:

$$\begin{aligned}\mathcal{R}(x)&=\frac{\beta^{\alpha}\,\Gamma(\alpha)}{x^{\alpha-1}}\exp\left(\frac{x}{\beta}\right)-\left(\exp\frac{x}{\beta}\right)\cdot\sum_{k=0}^{\infty}\frac{(-1)^k\,x^{k+1}}{\beta^k\,k!\,(k+\alpha)}\\&=\frac{\beta^{\alpha}\,\Gamma(\alpha)}{x^{\alpha-1}}\exp\left(\frac{x}{\beta}\right)-\sum_{r=0}^{\infty}\frac{x^r}{r!\,\beta^r}\cdot\sum_{k=0}^{\infty}\frac{(-1)^k\,x^{k+1}}{\beta^k\,k!\,(k+\alpha)}.\end{aligned} \tag{4}$$

If we put:

$$\sum_{r=0}^{\infty}\frac{x^r}{r!\,\beta^r}\sum_{k=0}^{\infty}\frac{(-1)^k\,x^{k+1}}{\beta^k\,k!\,(k+\alpha)}=\sum_{n=1}^{\infty}a_n\,x^n, \tag{5}$$

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as the conditions for the multiplying the series are satisfied, we have:

$$a_n = \frac{(-1)^{n-1}}{\beta^{n-1}(n-1)!} \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} \frac{1}{n-r-1+\alpha}. \tag{6}$$

From $\frac{1}{n-r-1+\alpha} = \int\limits_0^1 t^{n-r+\alpha-2}\,dt$, follows:

$$a_n = \frac{\Gamma(\alpha)}{\beta^{n-1}\Gamma(n+\alpha)} \tag{7}$$

and from (7) follows:

$$\mathcal{R}(x) = \frac{\beta^\alpha \Gamma(\alpha)}{x^{\alpha-1}} \exp\frac{x}{\beta} - \sum_{n=1}^{\infty} \frac{\Gamma(\alpha)}{\beta^{n-1}\Gamma(n+\alpha)} x^n, \quad x, \alpha, \beta > 0. \tag{8}$$

Formula (8) can be written in the form:

$$\mathcal{R}(x) = \frac{\beta^\alpha \Gamma(\alpha)}{x^{\alpha-1}} \exp\frac{x}{\beta} - \sum_{n=1}^{\infty} \frac{x^n}{\beta^{n-1}\alpha(\alpha+1)\cdots(\alpha+n-1)}. \tag{9}$$

3. The inequality $\Gamma(n+p) \leqq \Gamma(n+\alpha) \leqq \Gamma(n+p+1)$ is valid for $n>1$ $p \leqq \alpha \leqq p+1$, where p and n are natural numbers. From it follows:

$$\sum_{n=2} \frac{x^n}{\beta^{n-1}\Gamma(n+\alpha)} \geqq \frac{\beta^{p+1}}{x^p}\left(\sum_{n=0} \frac{x^n}{\beta^n n!} - \sum_{n=0}^{p+2} \frac{x^n}{\beta^n n!}\right), \tag{10}$$

$$\sum_{n=2}^{\infty} \frac{x^n}{\beta^{n-1}\Gamma(n+\alpha)} \leqq \frac{\beta^p}{x^{p-1}}\left(\sum_{n=0}^{\infty} \frac{x^n}{\beta^n n!} - \sum_{n=0}^{p+1} \frac{x^n}{\beta^n n!}\right). \tag{11}$$

From (10) and (11) follow the inequalities:

$$\frac{\beta^\alpha\Gamma(\alpha)}{x^{\alpha-1}} \exp\frac{x}{\beta} - \left\{\frac{x}{\alpha} + \frac{\beta^p\Gamma(\alpha)}{x^{p-1}}\left(\exp\frac{x}{\beta} - \sum_{n=0}^{p+1} \frac{x^n}{\beta^n n!}\right)\right\} \leqq \mathcal{R}(x) \tag{12}$$

$$\leqq \frac{\beta^\alpha\Gamma(\alpha)}{x^{\alpha-1}} \exp\frac{x}{\beta} - \left\{\frac{x}{\alpha} + \frac{\beta^{p+1}\Gamma(\alpha)}{x^p}\left(\exp\frac{x}{\beta} - \sum_{n=0}^{p+2} \frac{x^n}{\beta^n n!}\right)\right\}.$$

4. By the same method as in (12) we can find the bounds for remainders of the series in (8) and (9):

$$\frac{\Gamma(\alpha)\beta^{p+1}}{x^p}\left(\exp\frac{x}{\beta} - \sum_{n=0}^{p+N} \frac{x^n}{\beta^n n!}\right) \leqq \sum_{n=N}^{\infty} \frac{\Gamma(\alpha)}{\beta^{n-1}\Gamma(n+\alpha)} \tag{13}$$

$$\leqq \frac{\Gamma(\alpha)\beta^p}{x^p}\left(\exp\frac{x}{\beta} - \sum_{n=0}^{p+N-1} \frac{x^n}{\beta^n n!}\right).$$

5. We shall consider another approximation of the function (3). Let us write (3) as follows:

$$\mathcal{R}(x)=\beta^{\alpha}\,\Gamma(\alpha)\,x^{1-\alpha}\exp\frac{x}{\beta}-x^{1-\alpha}\beta^{\alpha}\left(\exp\frac{x}{\beta}\right)\left\{\Gamma(\alpha)-\int_{\frac{x}{\beta}}^{\infty}t^{\alpha-1}\exp(-t)\,dt\right\}.$$

For $\int_{u}^{\infty} t^{\alpha-1}\exp(-t)\,dt$ the continued fraction development is given in [3]:

$$\text{(14)}\qquad \int_{u}^{\infty} t^{\alpha-1}\exp(-t)\,dt=\exp(-u)u^{\alpha}\left\{\frac{1}{u+}\,\frac{1-\alpha}{1+}\,\frac{1}{u+}\,\frac{2-\alpha}{1+}\,\frac{2}{u+}\cdots\right\}$$

$$=\exp(-u)\cdot u^{\alpha}\,C(u).$$

From (14) follows: $\mathcal{R}(x)=x\,C\left(\frac{x}{\beta}\right)$, $R_1=\beta$, $R_2=\frac{x}{u+1-\alpha}$, $R_3=\frac{x(u+1)}{u^2+2u-\alpha u}$ etc.

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REFERENCES

1. D. S. Mitrinović (saradnik P. M. Vasić): *Analitičke nejednakosti.* Beograd 1970.
2. J. P. Mills: *Table of the ratio: area to bounding ordinate for any portion of the normal curve.* Biometrika **18** (1926), 395—4.
3. A. V. Boyd: *Bounds for Mills ratio for the type III population.* Ann. Math. Statistics **29**, № 3, 1958.
4. Des Ray: *On Mills ratio for the type III population.* Ann. Math. Statistics **24** (1953), 309—312.
5. Baikunth Nath: *An inequality for Mills ratio for the type III population.* Proc. Camb. Phil. Soc. **73** (1973), 351—353.
6. L. R. Shenton, J. A. Carpenter: *The Mills ratio and the probability integral for a Pearson Type IV distribution.* Biometrika (1964).

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MILLSOV KOLIČNIK ZA GAMA RASPODELU

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Polazeći od izraza (1), koji je u literaturi definisan za Gaussovu funkciju raspodele $F(x)$, definisan je količnik (3) za Γ-raspodelu.

U radu su dobijene aproksimacije količnika (3) konačnim sumama, kao i procena greške. Na kraju je data aproksimacija pomoću verižnih razlomaka.